[gex81] Poisson integrals for circle in complex analysis

Consider a function $f(z)$ which is analytic inside and on a circle of radius R. The Poisson integrals infer the values at $|z| < R$ from the values at $|z| = R$.

(a) Employ the Cauchy integral for a point z inside the circle and its inverse point $z_{\mathrm{inv}} = R^2/\bar{z}$ to derive the following relation:

$$f\big(re^{\imath\theta}\big) = \frac{1}{2\pi}\int_0^{2\pi} d\phi\, \frac{(R^2 - r^2)f(Re^{\imath\phi})}{R^2 - 2Rr\cos(\theta - \phi) + r^2}.$$

(b) Show that for given real and imaginary parts, $f(Re^{\imath\phi}) = u(R,\theta) + \imath v(R,\theta)$, the following relations hold:

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} d\phi\, \frac{(R^2 - r^2)u(R,\phi)}{R^2 - 2Rr\cos(\theta - \phi) + r^2},$$
$$v(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} d\phi\, \frac{(R^2 - r^2)v(R,\phi)}{R^2 - 2Rr\cos(\theta - \phi) + r^2}.$$

Solution: