[gex78] Green's theorem adapted to complex functions

Green's theorem – a special case of Stokes' theorem in vector analysis [gmd1-B] – expresses a relation between a line integral and a surface integral involving two real functions of two coordinates:

$$\oint_C [M(x,y)dx + N(x,y)dy] = \int_R dxdy \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right], \qquad (1)$$

where C is the simple perimeter loop of the compact region R. Here we consider a complex function $F(z,\bar{z}) = M(x,y) + \imath N(x,y)$, where $z = x + \imath y$, $\bar{z} = x - \imath y$ are conjugate complex variables. Show that the adaptation of Green's theorem to complex functions can be rendered as follows:

$$\oint_C dz\, F(z,\bar{z}) = 2\imath \int_R dxdy\, \frac{\partial F}{\partial \bar{z}}. \qquad (2)$$

Solution: