[gex71] Parabolic cylindrical coordinates

Parabolic cylindrical coordinates are best understood as a variation of the more familiar circular cylindrical coordinates. In both cases, the Cartesian z-component remains intact. Instead of replacing x, y by ρ, ϕ via $x = \rho\cos\phi,\ y = \rho\sin\phi$, which describe concentric circles, we replace them by u, v via

$$x = \frac{1}{2}(u^2 - v^2), \quad y = uv,$$

which describe two mutually orthogonal sets of confocal parabolas [gmd2].
(a) Use the prescription outlined in [gmd2] to determine the scale factors h_u, h_v, h_z for parabolic coordinates, which enables us to state all differential operators explicitly.
(b) Establish the relations,

$$u = \sqrt{2\rho}\,\cos\frac{\phi}{2}, \quad v = \sqrt{2\rho}\,\sin\frac{\phi}{2},$$

between the parabolic and circular versions of cylindrical coordinates.
(c) Prove that the curves $u = \text{const}$ and $v = \text{const}$ are indeed sets of confocal parabolas.
(d) Use the Mathematica command ParametricPlot to visualize these confocal parabolas similar to the plot shown in [gmd2].
Hint: For part (c) use the fact if the equation for a parabola is is written in the form $(y-h)^2 = 4a(x-k)$ then the vertex is at $y = h,\ x = k$ and the focus at $y = h,\ x = k + a$.

Solution: