[gex70] Laplacian operating on vector field

Consider a vector field stated in rectangular coordinates as follows:

$$\mathbf{V}_r(x,y,z) = x^2y^2\,\hat{\mathbf{i}} + y^3z\,\hat{\mathbf{j}} + x^2z^4\,\hat{\mathbf{k}}.$$

(a) For Cartesian coordinates, the Laplacian operating on a vector is equivalent to the vector sum of Laplacian operating on each coordinate treated as a scalar.

$$\nabla^2\mathbf{V}_r = \nabla^2 V_{rx}\,\hat{\mathbf{i}} + \nabla^2 V_{ry}\,\hat{\mathbf{j}} + \nabla^2 V_{rz}\,\hat{\mathbf{k}}.$$

Verify this relation. The Mathematica commands $\mathsf{Laplacian}[\mathsf{V_r}(\mathsf{x},\mathsf{y},\mathsf{z}),\{\mathsf{x},\mathsf{y},\mathsf{z}\},\text{"}\mathsf{Cartesian}\text{"}]$ works for vectors and scalars. This relation does not extend to curvilinear coordinates.
(b) Express the vector field $\mathbf{V}_r(x,y,z)$ as $V_c(\rho,\phi,z)$ in cylindrical coordinates and as $V_s(r,\theta,\phi)$ in spherical coordinates.
(c) Use the commands,

$$\mathsf{Laplacian}[\mathsf{V_c}(\rho,\phi,\mathsf{z}),\{\rho,\phi,\mathsf{z}\},\text{"}\mathsf{Cylindrical}\text{"}],\quad \mathsf{Laplacian}[\mathsf{V_s}(\mathsf{r},\theta,\phi),\{\mathsf{r},\theta,\phi\},\text{"}\mathsf{Spherical}\text{"}],$$

to calculate vectors representing the Laplacian of $\mathbf{V}$ in cylindrical and spherical coordinates.
(d) Show for all three sets of coordinates that the Laplacian operating on a vector (as executed by Mathematica) is equivalent to the following combination of the differential operators gradient, divergence, and curl:

$$\nabla^2\mathbf{V} = \nabla(\nabla\cdot\mathbf{V}) - \nabla\times(\nabla\times\mathbf{V}).$$

Use the corresponding Mathematica commands for this demonstration. The right-hand side of this identity is used as the definition of Laplacian applied to vectors in curvilinear coordinates.

Solution: