[gex67] Unit vectors of cylindrical coordinates

Start from the general relation (established in [gmd2-A]),

$$\frac{\partial \mathbf{s}}{\partial u_j} = \frac{\partial x_1}{\partial u_j}\,\hat{\mathbf{i}} + \frac{\partial x_2}{\partial u_j}\,\hat{\mathbf{j}} + \frac{\partial x_3}{\partial u_j}\,\hat{\mathbf{k}} = h_j \mathbf{e}_j,$$

between the Cartesian coordinates and any set of orthogonal curvilinear coordinates for the position vector $\mathbf{s} = x_1\,\hat{\mathbf{i}} + x_2\,\hat{\mathbf{j}} + x_3\,\hat{\mathbf{k}}$.
(a) Express the unit vectors $\mathbf{e}_1 = \hat{\boldsymbol{\rho}}$, $\mathbf{e}_2 = \hat{\boldsymbol{\phi}}$, $\mathbf{e}_3 = \hat{\mathbf{z}}$ of cylindrical coordinates as linear combinations of the unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$ of Cartesian coordinates.
(b) Verify that the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are an orthonormal set: $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$.
(c) Express the Cartesian unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$ as linear combinations of $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\phi}}$, $\hat{\mathbf{z}}$.
(d) Express the position vector $\mathbf{s} = x_1\,\hat{\mathbf{i}} + x_2\,\hat{\mathbf{j}} + x_3\,\hat{\mathbf{k}}$ in cylindrical coordinates by using the transformation relations for the x_i and the results of part (c) for $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$.
Hint: For part (a) take advantage of the transformation relations $x_i(u_1, u_2, u_3)$ and the scale factors h_j tabulated in [gmd2-A]. For part (c) use the Mathematica command LinearSolve or Inverse.

Solution: