[gex63] Work done by conservative force I

A vector field $\mathbf{F}(\mathbf{x})$ qualifies as a conservative force if the work,

$$W_{12} \doteq \int_{\mathbf{x}_1}^{\mathbf{x}_1} d\mathbf{x} \cdot \mathbf{F}(\mathbf{x}),$$

is path-independent. A force expressible as the gradient of a potential energy, $\mathbf{F}(\mathbf{x}) = -\nabla U(\mathbf{x})$, is conservative. Conservative forces are irrotational, $\nabla \times \mathbf{F}(\mathbf{x}) = 0$.
(a) Show that the following force is conservative:

$$\mathbf{F}(\mathbf{x}) = (ay + 2cxyz)\hat{\mathbf{i}} + (ax + bz^2 + cx^2z)\hat{\mathbf{j}} + (2byz + cx^2y)\hat{\mathbf{k}}.$$

In this exercise, a, b, c are constants and coordinates are dimensionless.
(b) Determine the potential energy $U(\mathbf{x})$ associated with this force such that $U(\mathbf{0}) = 0$.
(c) Consider a particle being moved quasi-statically between position $\mathbf{x}_1 = (1,1,1)$ and position $\mathbf{x}_2 = (2,2,2)$ along the following path:

$$(1,1,1) \quad \xrightarrow{W_x} \quad (2,1,1) \quad \xrightarrow{W_y} \quad (2,2,1) \quad \xrightarrow{W_z} \quad (2,2,2).$$

Calculate the work along each leg and the sum $W_{12} = W_x + W_y + W_z$.
(d) Show that $W_{12} = -[U(\mathbf{x}_2) - U(\mathbf{x}_1)]$.

Solution: