[gex6] First-order ODE: separation of variables III

Water flows into a cone-shaped container at a constant rate (volume per unit time) and evaporates at a rate proportional to the free surface area:

$$\frac{dV}{dt} = a - bV^{2/3}.$$

(a) Find the SI units of the constants a and b.
(b) Determine the equilibrium position of the water level, expressed as the volume V_{eq} at which the two processes are in balance.
(c) Determine whether or not that stationary state is asymptotically stable by investigating (perturbatively) the time evolution of a small deviation from stationarity.
(d) Transform the ODE given above into the parameterless ODE,

$$\frac{dx}{d\bar{t}} = 1 - x^{2/3}, \quad x \doteq \frac{V}{V_{eq}}, \quad \bar{t} = \frac{ta}{V_{eq}}.$$

(e) Use the Mathematica command NDSolve to determine the numerical solution for initial condition $x(0) = 0$ and range $0 < \bar{t} < 8$. Then plot that curve using the command Plot.
(f) Find an analytic solution of the ODE by variable substition $x = y^3$, separation of variables, and integration. The immediate result is a function $\bar{t}(y)$. Use the command ParametricPlot to poduce a curve in the same format as in part (e). Show that the results are equivalent.

Solution: