

[gex46] Matrix operations VII: system of linear equations

Consider the system of inhomogeneous linear equations,

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 &= r_1, \\a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 &= r_2, \\a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 &= r_3, \\a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 &= r_4,\end{aligned}$$

with coefficients, inhomogeneities, and solution expressed in matrix form as follows:

$$\mathbf{A} = \begin{pmatrix} 4 & 1 & -3 & 2 \\ 3 & -1 & 2 & -4 \\ 3 & -1 & 4 & -2 \\ 1 & -2 & -4 & -3 \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

- Verify that $\text{Det}[\mathbf{A}] \neq 0$, which ensures a unique solution.
- Use the Mathematica command `LinearSolve` to determine the solution \mathbf{X} .
- Determine the inverse matrix \mathbf{A}^{-1} , which is guaranteed to exist when $\text{Det}[\mathbf{A}] \neq 0$, and reproduce the solution via $\mathbf{X} = \mathbf{A}^{-1}\mathbf{R}$.
- Replace the inhomogeneity \mathbf{R} by the null vector $\mathbf{0}$ and show that `LinearSolve` correctly predicts an identically vanishing \mathbf{X} as the only solution.

Replace the coefficient matrix \mathbf{A} by coefficient matrix \mathbf{B} with $b_{33} = 2$, $b_{34} = -4$ and all other coefficients unchanged. The matrix \mathbf{B} then has two identical rows of elements.

- Confirm that $\text{Det}[\mathbf{B}] = 0$ and that `LinearSolve` correctly states that $\mathbf{B}\mathbf{X} = \mathbf{R}$ has no solution.
- Confirm that `LinearSolve` predicts an identically vanishing \mathbf{X} as the solution of the homogeneous equations, $\mathbf{B}\mathbf{X} = \mathbf{0}$, which is correct but incomplete.
- Use the `Solve` command to find an infinity of additional solutions of $\mathbf{B}\mathbf{X} = \mathbf{0}$. Express these solutions in the form x_1, x_2, x_3 expressed as functions of x_4 .

Create a Mathematica notebook to carry out these tasks.

Solution: