[gex45] Matrix operations VI: determinant of matrix products

Determinants exist for square matrices only. The (non-commutative) product of two $n \times n$ square matrices $\mathbf{S}$, $\mathbf{T}$ is an $n \times n$ square matrix again. The determinant of the product matrix is the product of determinants: $\mathrm{Det}[\mathbf{ST}] = \mathrm{Det}[\mathbf{TS}] = \mathrm{Det}[\mathbf{S}]\,\mathrm{Det}[\mathbf{T}]$.
The product of an $n \times m$ matrix $\mathbf{A}$ and an $m \times n$ matrix $\mathbf{B}$ produces either the $n \times n$ square matrix $\mathbf{AB}$ or the $m \times m$ square matrix $\mathbf{BA}$. Determinants exist for both product matrices.
Consider the the two rectangular matrices,

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}.$$

Demonstrate that $\mathrm{Det}[\mathbf{BA}] = 0$ for arbitrary values of the matrix elements. By contrast, $\mathrm{Det}[\mathbf{AB}]$ is non-vanishing, in general. Create a Mathematica notebook to carry out these tasks.

Solution: