

[gex33] Common identities used electrostatics and elsewhere

The following two identities involving derivatives of the quantity $1/|\mathbf{x} - \mathbf{x}'|$, which represents the inverse magnitude of a distance vector in Euclidean space, have proven immensely useful in the processing of vector fields:

$$\nabla \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) = -\frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3}, \quad \nabla^2 \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) = -4\pi\delta(\mathbf{x} - \mathbf{x}').$$

- (a) Prove the first identity by carrying out the partial derivatives of the gradient operator in Cartesian coordinates.
- (b) Prove the second identity in two steps. (i) Carry out the Laplacian in spherical coordinates (see [gmd2]) for $|\mathbf{x} - \mathbf{x}'| \neq 0$. (ii) Perform a volume integral of $1/|\mathbf{x} - \mathbf{x}'|$ over a sphere centered at $\mathbf{x} - \mathbf{x}' = 0$ using Gauss's theorem.

Solution: