# [gex25] First-order ODE: Clairaut type

Consider the nonlinear 1st-order ODE,

$$(y - xy')^2 = y'^2 + 1.$$

(a) Show that it is of the Clairaut type as described in [gmd10] and can be rewritten in the form stated with obvious general solutions:

$$y = xp \pm \sqrt{p^2 + 1}, \quad p \doteq \frac{dy}{dx} \quad \Rightarrow \ y(x) = cx \pm \sqrt{c^2 + 1}.$$

(b) Show that the Mathematica DSolve command returns exactly these two one-parameter general solutions of the original ODE.

(c) Plot both general solutions with the parameter systematically varied in the same diagram.

(d) The emerging pattern suggest the existence of a singular solution – a curve which touches all particular solutions inferred from the general solution. Guess the shape of the singular solution and add it to the plot using the ContourPlot command.

(e) In order to find the singular solution analytically, we convert the ODE in the Clairaut form into a 1st-order ODE for the variable $p$. This is the convertibility method described in [gmd10]. Show that the resulting ODE can be brought into the form,

$$\frac{dp}{dx}\left(x \pm \frac{p}{\sqrt{p^2 + 1}}\right) = 0.$$

One of the two factors must vanish. The first factor vanishes for the general solution already found. Therefore, we infer the singular solution by setting the second factor equal to zero. Show that this condition leads to the implicit solution you already guessed in part (d).

**Solution:**