[gex2] Recurrence relation for Gamma function

The Gamma function $\Gamma(z)$ with $z \in \mathbb{C}$ is commonly introduced by its integral representation,

$$\Gamma(z) \doteq \int_0^\infty dt e^{-t} t^{z-1} \quad : \Re[z] > 0, \tag{1}$$

or by the (more general) Weierstrass product representation,

$$\Gamma(z) \doteq \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \frac{e^{z/n}}{1 + z/n} \quad : z \notin \{0, -1, -2, \ldots\}, \tag{2}$$

where the Euler constant is

$$\gamma \doteq \lim_{N\to\infty} \left(\sum_{n=1}^{N} \frac{1}{n} - \ln N \right) = 0.57721\ldots \tag{3}$$

(a) Show that $\Gamma(1) = 1$ follows from (1) and also from (2).
(b) Infer the relation, $\Gamma(z+1) = z\Gamma(z)$, first using (1) only and then using (2) only with (3).

Solution: