

[gex17] First-order ODE: convertibility

If the goal is to find all solutions $y(x)$ of the nonlinear 1st-order ODE,

$$xy'^2 + 2xy' - y = 0,$$

explicitly or implicitly, it is reasonable to start with the DSolve comand of Mathematica. Mathematica will return two explicit solutions $y(x)$, each with one integration constant. These two functions are, as it will turn out, branches of the two-valued general solution. However, there is also a singular solution, which the DSolve does not return.

- (a) Record the solutions returned by the DSolve command in the most simplified rendition.
- (b) The ODE is convertible as described in [gmd10-A]. Show how you transform it into

$$p(p + 1) + 2x(p + 1)p' = 0, \quad p = y'.$$

- (c) Taking note that two cases must be distinguished, find the general solution for $p \neq -1$ in implicit form via separation of variables.
- (d) Infer an implicit solution, $f(x, y, c) = 0$, for the original ODE.
- (e) Use ContourPlot with $0 < x < 2$, $-2 < y < 2$ to graphically represent the functional dependence between y and x of this solution for various values of the integration constant.
- (e) Show that this implicit solution is equivalent to the explicit solution returned by DSolve.
- (f) Return the original ODE and set $p = y' = -1$. It produces a singular solution, which has no integration constant.
- (g) Add the singular solution to the plot of the general solution to show compatibility.

Solution: