[gex117] Radial and azimuthal fields in a plane

Consider a radial vector field $\mathbf{R}$ and an azimuthal vector field $\mathbf{P}$ of uniform strength in a plane,

$$\mathbf{R} = a\,\hat{\boldsymbol{\rho}} = a\cos\phi\,\hat{\mathbf{i}} + a\sin\phi\,\hat{\mathbf{j}}, \qquad \mathbf{P} = b\,\hat{\boldsymbol{\phi}} = -b\sin\phi\,\hat{\mathbf{i}} + b\cos\phi\,\hat{\mathbf{j}},$$

with constant a, b. The last expression transforms each field to rectangular coordinates [gmd2]. The transformation between rectangular and polar coordinates,

$$x = \rho\cos\phi, \quad y = \rho\sin\phi,$$

is nonlinear and orthogonal.
(a) If we insist that $R^1 = a,\ R^2 = 0$ and $P^1 = 0,\ R^2 = b$ are the elements of contravariant rank-1 tensors in polar coordinates, what are their elements $\bar{R}^1,\ \bar{R}^2$ and $\bar{P}^1,\ \bar{P}^2$ in rectangular coordinates?
(b) If instead we assume that $R_1 = a,\ R_2 = 0$ and $P_1 = 0,\ R_2 = b$ are the elements of covariant rank-1 tensors in polar coordinates, what are their elements $\bar{R}_1,\ \bar{R}_2$ and $\bar{P}_1,\ \bar{P}_2$ in rectangular coordinates?
(c) Use The results of parts (a) and (b) to recover the original tensors via the inverse transition from rectangular to polar coordinates.
Hint: Use the elements of the Jacobian and its inverse determined in [gex112].

Solution: