[gex113] Uniform vector field in plane made into a tensor

Consider the vector field $\mathbf{A} \doteq (A_x, A_y) = a\,\hat{\mathbf{i}} + b\,\hat{\mathbf{j}}$ in $\mathbb{R}^2$ with constant $a,\ b$. Finding the components of this vector field in polar coordinates is straightforward and a topic investigated in [gmd2]:

$$\mathbf{A} \doteq (A_\rho, A_\phi) = (a\cos\phi + b\sin\phi)\hat{\boldsymbol{\rho}} + (b\cos\phi - a\sin\phi)\hat{\boldsymbol{\phi}}.$$

The transformation between rectangular and polar coordinates,

$$x = \rho\cos\phi, \quad y = \rho\sin\phi,$$

is nonlinear and orthogonal.
(a) If we insist that $A^1 = a,\ A^2 = b$ are the elements of a contravariant rank-1 tensor in rectangular coordinates, what are its elements $\bar{A}^1,\ \bar{A}^2$ in polar coordinates?
(b) If instead we assume that $A_1 = a,\ A_2 = b$ are the elements of a covariant rank-1 tensor in rectangular coordinates, what are its elements $\bar{A}_1,\ \bar{A}_2$ in polar coordinates?
(c) Use The results of parts (a) and (b) to recover the original tensors via the inverse transition from polar to rectangular coordinates.
Hint: Use the elements of the Jacobian and its inverse determined in [gex112].

Solution: