[gex11] Plane-pendulum rotations

A point mass m is constrained by a massless rod to move in a vertical circle of radius l in a uniform gravitational field g. Rotational motion is realized for energies in the range $E > 2mgl$. The dynamical variable of choice is the angle $\theta(t)$ of the rod away from the downward orientation.
(a) Show that the time evolution of the angle expressed as a Jacobi elliptic function and the period of rotation expressed as a complete elliptic integral are

$$\theta(t) = 2\arcsin\Big(\mathrm{sn}(\omega_0 t/\kappa, \kappa)\Big), \quad \tau = \frac{2\kappa}{\omega_0}\,\mathrm{K}(\kappa); \quad \omega_0 = \sqrt{\frac{g}{l}}, \quad \kappa = \sqrt{\frac{2mgl}{E}}$$

(b) Work out simplified asymptotic expressions for the case of fast rotations $\kappa \ll 1$ and for the (non-periodic) separatrix motion in the limit $\kappa \to 1$.
(c) Find an analytic expression for the angular velocity $\dot{\theta}(t)$.
(d) Design graphical representations for $\theta(t)$ and $\dot{\theta}(t)$ to illustrate the impact of gravity on the pendulum rotation.

Solution: