[gex10] Plane-pendulum oscillations

A point mass m is constrained by a massless rod to move in a vertical circle of radius $l$in a uniform gravitational field g. Oscillatory motion is realized for energies in the range $0 < E < 2mgl$. The dynamical variable of choice is the angle $\theta(t)$ of the rod away from the downward orientation.
(a) Show that the time evolution of the angle expressed as a Jacobi elliptic function and the period of oscillation expressed as a complete elliptic integral are

$$\theta(t) = 2\arcsin\Big(k\,\mathrm{sn}(\omega_0 t, k)\Big), \quad \tau = \frac{4}{\omega_0}\,\mathrm{K}(k); \quad \omega_0 = \sqrt{\frac{g}{l}}, \quad k = \sqrt{\frac{E}{2mgl}}$$

(b) Recover the harmonic oscillations for $k \ll 1$ and the (non-periodic) separatrix motion in the limit $k \to 1$.
(c) Find an analytic expression for the angular velocity $\dot{\theta}(t)$.
(d) Design graphical representations for $\theta(t)$ and $\dot{\theta}(t)$ to illustrate the anharmonic effects in the pendulum oscillations.

Solution: