[gex1] Representations of the Dirac delta function I

Show that the two functions,

$$\Delta_1(x, w) \doteq \frac{1}{\sqrt{\pi w}} e^{-(x/w)^2}, \quad \Delta_2(x, w) \doteq \frac{\Theta(x - w/2) - \Theta(x + w/2)}{w},$$

are representations of the Dirac delta function $\delta(x)$ in the limit $w \to 0$ by showing that the two attributes stated in [lln4],

$$\delta(x) = \begin{cases} 0 & : x \neq 0, \\ \infty & : x = 0, \end{cases} \qquad \int_{-\infty}^{+\infty} dx\, \delta(x) = 1.$$

are satisfied. Then calculate (either numerically or analytically) the integrals

$$I_1(w) = \int_{-\infty}^{+\infty} dx\, f(x)\Delta_1(x - 2, w), \quad I_2(w) = \int_{-\infty}^{+\infty} dx\, f(x)\Delta_2(x - 2, w), \quad f(x) = x^2,$$

for small values of w and show that the results approach the value $f(2) = 4$ as would be the case if $\delta(x - 2)$ were substituted for $\Delta_1(x, w)$ or $\Delta_1(x, w)$ in the two integrals.

Solution: