Laplace Transform [gmd8A]

The Laplace transform in its most elementary manifestation is a linear integral operator $\mathcal{L}$ applied to a real function function $f(t)$ of a real variable t, yielding a real function $F(s)$ of a real variable s:

$$\mathcal{L}\{f(t)\} \doteq \int_0^\infty dt\, e^{-st} f(t) = F(s).$$

Linearity: $\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1\mathcal{L}\{f_1(t)\} + c_2\mathcal{L}\{f_2(t)\}$.

For some functions $f(t)$, the integral does not converge. The Laplace transform does not exist in that case. For some other functions $f(t)$, the range of convergence is restricted to $s \geq s_0 > 0$.

Sufficient existence condition: If $f(t)$ is piecewise continuous in every finite interval $0 \leq t \leq T$ and is of exponential order for $t \geq T$ [i.e. satisfies $|f(t)| \leq M e^{\alpha t}$], then its Laplace transform exists for $s > \alpha$.

Inverse Laplace transform: $\mathcal{L}^{-1}\{F(s)\} = f(t)$.

The operator $\mathcal{L}^{-1}$ is also linear – an attribute to be used from here on – and it is also an integral operator – an attribute to be demonstrated later.

The Laplace transform, if it exists, is unique in the forward direction, but not quite unique in the reverse direction. However, two functions with equal Laplace transform can only differ in isolated points.

Laplace transforms (and, of course, their inverse) for elementary functions and combinations thereof are available in compilations and are now readily generated computationally.

It is useful to remember some of the most elementary cases:

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \qquad \mathcal{L}\{e^{-at}\} = \frac{1}{s+a}, \qquad \mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2},$$
$$\mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2}, \quad \mathcal{L}\{\sinh(at)\} = \frac{a}{s^2-a^2}, \quad \mathcal{L}\{\cosh(at)\} = \frac{s}{s^2-a^2}.$$

For the hyperbolic functions, convergence requires $s > |a|$.

The Laplace transform of the derivative $f'(t)$ is simply related to that of $f(t)$ via an integration by parts (see [gex62] for higher derivatives):

$$\mathcal{L}\{f'(t)\} = \int_0^\infty dt\, e^{-st} f'(t) = \Big[e^{-st} f(t)\Big]_0^\infty - \int_0^\infty dt(-s) e^{-st} f(t)$$
$$= -f(0) + s\int_0^\infty dt\, e^{-st} f(t) = \mathcal{L}\{f(t)\} - f(0).$$

Here we test the rule for derivatives with the Dirac delta function $\delta(t)$, which, in a generalized sense, is the derivative of the Heaviside step function $\Theta(t)$:

$$\mathcal{L}\{\Theta(t-a)\} = \frac{e^{-as}}{s}, \quad \mathcal{L}\{\delta(t-a)\} = e^{-as},$$
$$\mathcal{L}\{\Theta'(t-a)\} = s\mathcal{L}\{\Theta(t-a)\} - \Theta(-a) = e^{-sa} - 0 = \mathcal{L}\{\delta(t-a)\}.$$

Compilation of a few useful facts (or theorems) about the Laplace transform, $\mathcal{L}\{f(t)\} = F(s)$, and its inverse, $\mathcal{L}^{-1}\{F(s)\} = f(t)$.

- Translation:

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a), \quad \mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t).$$

- Step function:

$$\mathcal{L}\{\Theta(t-a)f(t-a) = e^{-as}F(s), \quad \mathcal{L}^{-1}\{e^{-as}F(s)\} = \Theta(t-a)f(t-a).$$

- Scaling:

$$\mathcal{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right), \quad \mathcal{L}^{-1}\left\{\frac{1}{a}F\left(\frac{s}{a}\right)\right\} = f(at).$$

- Moments:

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s), \quad \mathcal{L}^{-1}\left\{\frac{d^n}{ds^n}F(s)\right\} = (-t)^n f(t).$$

- Integral:

$$\mathcal{L}\left\{\int_0^t du\, f(u)\right\} = \frac{F(s)}{s}, \quad \mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t du\, f(u).$$

- Periodic functions:

$$f(t+a) = f(t) \quad \Rightarrow \mathcal{L}\{f(t)\} = [1-e^{-as}]^{-1}\int_0^a dt\, e^{-st}f(t).$$

- Limit theorem:

$$\mathcal{L}\{f(t)\} = F(s), \quad \lim_{t\to 0}\frac{f(t)}{t} \quad \Rightarrow \mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty du\, F(u).$$

Convolution theorem:

If $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$ hold, then the convolution integral Laplace transforms into a product of functions:

$$\mathcal{L}\{f \circ g\} \doteq \mathcal{L}\left\{\int_0^t du\, f(u)g(t-u)\right\} = F(s)G(s).$$

A product of functions inverse-Laplace transforms into a convolution integral:

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t du\, f(u)g(t-u).$$

The convolution integral is commutative, associative, and distributive:

$$f \circ g = g \circ f, \quad f \circ (g \circ h) = (f \circ g) \circ h, \quad f \circ (g + h) = f \circ g + f \circ h.$$