Integral Transforms [gmd8]

The role of integral transforms as an analytic tool in the processing of functions is akin to that of logarithms in the processing of numbers. The domain of applications include differential and integral equations.

The general form of an integral transform can be cast as follows:

$$F(s) \doteq \int_{-\infty}^{\infty} dt\, K(s,t) f(t),$$

- ▷ $f(t)$: original function,
- ▷ t: original variable,
- ▷ $F(s)$: transformed function,
- ▷ s: transformed variable,
- ▷ $K(s,t)$ kernel of integral transform.

A unique function $F(s)$ exists if $K(s,t)$ and $f(t)$ satisfy certain conditions.

- Fourier transform: $K(s,t) \doteq \dfrac{1}{\sqrt{2\pi}}\, e^{\imath st}$.
  Integral transform with the widest range of applications.
- Laplace transform: $K(s,t) \doteq e^{-st}\theta(t)$.
  Integral transform tailored for initial value problems.
- Hankel transform: $K(s,t) \doteq t\mathrm{J}_\nu(st)\theta(t)$.
  Applications to boundary value problems in cylindrical coordinates. $\mathrm{J}_\nu(x)$ is a Bessel function.
- Mellin transform: $K(s,t) \doteq t^{s-1}\theta(t)$.
  Applications to boundary value problems with wedge-shaped regions.

The domain of t is restricted by a vanishing kernel in some transforms. The Laplace transform, for example, has a semi-infinite domain. Transforms on finite domains are named *finite* integral transforms.

The usefulness of integral transforms relies on the existence and uniqueness of inverse transforms. The inverse Fourier transform is has the simplest form:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} ds\, e^{-\imath st} F(s).$$

In general, inverse transforms are more complicated.

Further integral transforms include the *Hilbert* transform and the *Sturm-Liouville* transform. Discrete Fourier transforms (Fourier series) and discrete Laplace transforms (Z transforms) are also important.