Complex Analysis II [gmd7-B]

Line integrals:

Consider a curve C of finite length[1] in the complex plane and a complex function $f(z,\bar{z}) = u(x,y) + \imath v(x,y)$ which is continuous along the curve C.

Continuous functions are line-integrable and the line integral is constructed as a limit process:

$$\sum_{k=1}^{n} f(\zeta_k,\bar{\zeta}_k)(z_k - z_{k-1}) \stackrel{n\to\infty}{\longrightarrow} \int_C dz f(z,\bar{z}).$$

Here the z_k are (roughly equidistant) points on the curve and the ζ_k are roughly midway between z_{k-1} and z_k.[2] The endpoints z_0 and z_n of the curve may or may not coincide.

Relation to line integrals examined in vector analysis [gmd1-B]:

$$\begin{aligned}\int_C dz f(z,\bar{z}) &= \int_C (dx + \imath dy)[u(x,y) + \imath v(x,y)] \\ &= \int_C [dx\, u(x,y) - dy\, v(x,y)] + \imath \int_C [dx\, v(x,y) + dy\, u(x,y)].\end{aligned}$$

Parametrization of the curve:

$$x = \phi(t),\ y = \psi(t) \quad \Rightarrow\ dx = \phi'(t)dt,\ dy = \psi'(t)dt.$$

$$\begin{aligned}\Rightarrow \quad \int_C dz f(z,\bar{z}) &= \int_{t_i}^{t_f} dt \Big[u\big(\phi(t),\psi(t)\big)\phi'(t) - v\big(\phi(t),\psi(t)\big)\psi'(t)\Big] \\ &\quad + \imath \int_{t_i}^{t_f} dt \Big[v\big(\phi(t),\psi(t)\big)\phi'(t) + u\big(\phi(t),\psi(t)\big)\psi'(t)\Big].\end{aligned}$$

A simple curves in the complex plane does not intersect itself. *Jordan curves* are simple closed curves.[3] Regions in the complex plane can be simply or multiply connected. Simply connected regions are without holes.

All Jordan curves in a simply connected region can be shrunk to a point without leaving it. All Jordan curves divide the complex plane into an inside and an outside. The interior region is simply connected.

Integrating a complex function along a Jordan curve in a counterclockwise sense is, by convention, the positive (negative) sense for the interior (exterior) region. The relevant region is on the left (right).

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[1] Such curves are named *rectifiable*.

[2] The necessary conditions are more relaxed.

[3] Jordan curves with fractal structure can have infinite length.

Cauchy's theorem:

Green's theorem – a special case of Stokes' theorem examined in vector analysis [gmd1-B] – expresses a relation between a line integral and a surface integral involving two real functions of two coordinates.

When adapted to a complex function as worked out in [gex78] it can be rendered as follows:

$$\oint_C dz\, F(z,\bar{z}) = 2\imath \int_R dxdy\, \frac{\partial F}{\partial \bar{z}},$$

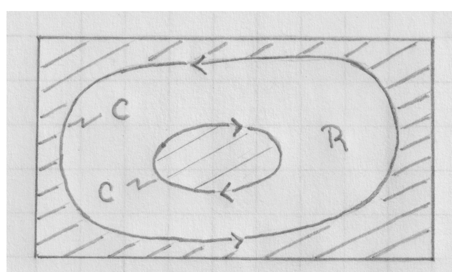
where the curve C (named contour) is the boundary of region R.

Recall that for analytic functions $F(z,\bar{z}) = f(z)$ we have $\partial F/\partial \bar{z} \equiv 0$, which leads to Cauchy's theorem,

$$\oint_C dz\, f(z) = 0.$$

Cauchy's theorem is valid for any function $f(z)$ which is analytic in a simply or multiply connected region R of which C is the boundary.

The boundary of a multiply connected region has multiple parts, which must be integrated in a consistent (positive or negative) sense.



Cauchy's theorem remains valid (under mild and obvious restrictions) for curves inside region R.

The converse of Cauchy's theorem (Morera's theorem) states that if the line integral of a function $f(z)$ vanishes for all curves in a simply connected region R, then $f(z)$ is analytic. A generalization to multiply connected R exists.

Cauchy's theorem guarantees the existence of indefinite integrals for analytic functions:

$$F(z) = \int dz f(z) \quad \Leftrightarrow \quad f(z) = \frac{dF}{dz}.$$

Cauchy integrals:

Consider a function $f(z)$ which is analytic in a region R bounded by a simple closed curve C. It is then possible to infer from Cauchy's theorem what is known as the Cauchy integral,

$$f(a) = \frac{1}{2\pi\imath} \oint_C dz \, \frac{f(z)}{z-a},$$

with $a \in R$ and the contour integral traversed in the positive (ccw) sense.

- ▷ Cauchy's theorem permits the replacement of C by a tiny circle of radius ϵ centered at a without changing the value of the integral.
- ▷ $z - a = \epsilon e^{\imath\phi}, \quad dz = \imath\epsilon e^{\imath\phi} d\phi.$
- ▷ Convert integral:

$$\oint_C dz \, \frac{f(z)}{z-a} = \int_0^{2\pi} d\phi \frac{\imath\epsilon e^{\imath\phi} f(a+\epsilon e^{\imath\phi})}{\epsilon e^{\imath\phi}} = \imath \int_0^{2\pi} d\phi f(a+\epsilon e^{\imath\phi}).$$

- ▷ Shrink radius to zero:

$$\xrightarrow{\epsilon\to 0} \quad \imath \int_0^{2\pi} d\phi \, f(a) = 2\pi\imath f(a).$$

Cauchy integrals also exist for the n^{th} derivative of the function $f(z)$:

$$f^{(n)}(a) = \frac{n!}{2\pi\imath} \oint_C dz \, \frac{f(z)}{(z-a)^{n+1}} \qquad : n = 0, 1, 2, \ldots$$

Taking derivatives with respect to a on the left and (under the contour integral) on the right is an application of Leibniz's rule.

Some consequences of Cauchy integrals are stated in the following, for the most part without proof.

Existence of derivatives:

- If $f(z)$ is known on C then the values of $f(z)$ and all its derivatives are determined at all points inside C.
- A function $f(z)$ is analytic in R if its first derivative exists. All higher derivatives then exist as well. No such can conclusion can be drawn for functions of real variables.

Cauchy inequality and Liouville theorem:

– *Cauchy's inequality*: If $f(z)$ is analytic inside a circle of radius r around $z = a$ and $|f(z)| < M$ on circle, the following inequality holds:

$$|f^{(n)}(a)| \leq \frac{Mn!}{r^n} \quad : \ n = 0, 1, 2, \ldots$$

– *Liouville's theorem*: If $f(z)$ is analytic and bounded ($|f(z)| < M < \infty$) throughout the complex plane, then $f(z) = \text{const}$.

  ▷ If Cauchy's inequality for $n = 1$ holds at all points for unlimited radius r, we can conclude that $f'(z) \equiv 0$. Hence $f(z) = \text{const}$.

Fundamental theorem of algebra:

Every polynomial equation,

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n = 0 \quad : \ n \geq 1, \quad a_n \neq 0,$$

has at least one root (in fact, exactly n roots).

▷ If analytic $P(z)$ had no roots, then $1/P(z)$ would be bounded and analytic, i.e. a constant, in contradiction to the premise.

▷ If $P(z)$ has at least one root, z_1, it can be written in the form $P(z) = (z - z_1)Q(z)$, where $Q(z)$ is a polynomial of degree $n - 1$, to which the theorem applies as well.

This theorem is an important ingredient of a similarly named theorem of linear ODEs [gam8].

Mean-value and maximum-modulus theorems:

– *Gauss's mean-value theorem*: The mean value of an analytic function on a circle is equal to its value at the center:

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} d\phi \, f\big(a + re^{\imath\phi}\big).$$

  ▷ Application of Cauchy integral to circular path, $z = a + re^{\imath\phi}$:

$$f(a) = \frac{1}{2\pi\imath} \int_0^{2\pi} d\phi \, \frac{\imath re^{\imath\phi} f(a + re^{\imath\phi})}{re^{\imath\phi}} = \frac{1}{2\pi} \int_0^{2\pi} d\phi \, f\big(a + re^{\imath\phi}\big).$$

– *Maximum-modulus theorem*: A function $f(z)$ which is analytic inside and on a simple closed curve C cannot have a maximum in modulus $|f(z)|$ inside the curve. The maximum is always located on the curve.

  ▷ A function $f(z)$ with a maximum modulus at a point a inside C would violate the mean-value theorem.

  ▷ A corresponding minimum-modulus theorem exists on the condition that $f(z) \neq 0$ inside C.

Argument theorem and Rouché's lemma:

– *Argument theorem*: If $f(z)$ is analytic inside and on a simple closed curve C except for a pole of order p and a zero of order n, we have

$$\frac{1}{2\pi\imath} \oint_C dz\, \frac{f'(z)}{f(z)} = n - p.$$

  ▷ A pole of order p [zero of order n] means that we can express $f(z)$ in the form,

$$f(z) = \frac{F(z)}{(z - z_{\text{pole}})^p}, \quad [f(z) = (z - z_{\text{zero}})^n G(z)],$$

  where $F(z)$ and $G(z)$ are analytic and nonzero inside and an C.

  ▷ The theorem (proven in [gex80]) is readily generalized to functions with several poles and zeros of varying order. The n and p on the right-hand side must then be replaced by sums of n_i and p_j representing orders of different zeros and poles, respectively.

– *Rouché's lemma*: If $f(z)$ and $g(z)$ are analytic inside and on a simple closed curve C and if $|g(z)| < |f(z)|$ on C, then adding $g(z)$ to $f(z)$ does not change the number of zeros inside.

  ▷ The value of this theorem comes into play in the proof of other theorems, e.g. the fundamental theorem of algebra.

Poisson integrals:

– *Poisson integrals for circle*: For a function $f(z)$ which is analytic both inside and on a circle of radius R the values at $|z| < R$ can be determined from the values at $|z| = R$:

$$f\left(re^{\imath\theta}\right) = \frac{1}{2\pi}\int_0^{2\pi} d\phi\, \frac{(R^2 - r^2) f(Re^{\imath\phi})}{R^2 - 2Rr\cos(\theta - \phi) + r^2} \quad : \ r < R.$$

For given real and imaginary parts, $f(Re^{\imath\phi}) = u(R,\theta) + \imath v(R,\theta)$:

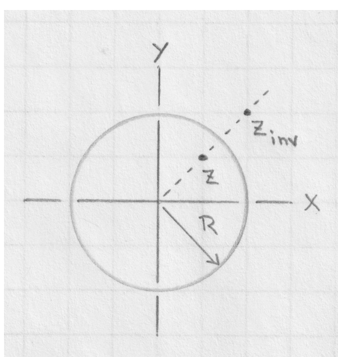
$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} d\phi\, \frac{(R^2 - r^2) u(R,\phi)}{R^2 - 2Rr\cos(\theta - \phi) + r^2} \quad : \ r < R,$$

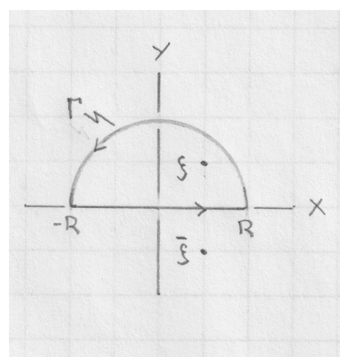
$$v(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} d\phi\, \frac{(R^2 - r^2) v(R,\phi)}{R^2 - 2Rr\cos(\theta - \phi) + r^2} \quad : \ r < R.$$

▷ The proof of the Poisson integrals [gex81] uses the variable $z_{\rm inv}$ which is inverse to z with respect to the circle of radius R:

$$z = re^{\imath\phi}, \quad \bar{z} = re^{-\imath\phi} \quad \Rightarrow \ z_{\rm inv} = \frac{R^2}{\bar{z}} = \frac{R^2}{r}\, e^{\imath\phi}.$$

The points z and $z_{\rm inv}$ have equal arguments. Their moduli are related by $|z|/R = R/|z_{\rm inv}|$. If z is inside then $z_{\rm inv}$ is outside.





– *Poisson integrals for half plane*: For a function $f(\zeta)$ which is analytic and bounded for $\Im[\zeta] \geq 0$ the values at $\Im[\zeta] > 0$ can be determined from the values at $\Im[\zeta] = 0$ [gex83]:

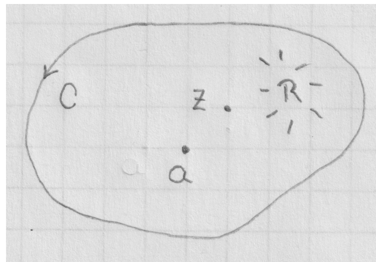
$$f(\zeta) = \frac{1}{\pi}\int_{-\infty}^{\infty} dx\, \frac{\eta f(x)}{(x - \xi)^2 + \eta^2} \quad : \ \zeta = \xi + \imath\eta, \quad \eta > 0.$$

Setting $f(\zeta) = u(\xi,\eta) + \imath v(\xi,\eta)$, we can write,

$$u(\xi,\eta) = \frac{1}{\pi}\int_{-\infty}^{\infty} dx\, \frac{\eta u(x,0)}{(x - \xi)^2 + \eta^2}, \quad v(\xi,\eta) = \frac{1}{\pi}\int_{-\infty}^{\infty} dx\, \frac{\eta v(x,0)}{(x - \xi)^2 + \eta^2}.$$

Taylor series:

Consider a simply connected region R bounded by the curve C.



If $f(z)$ is analytic inside R and on C, then Taylor's theorem states that the following series converges for all points z and a inside R:

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2}(z-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(z-a)^n + \cdots .$$

▷ A proof (not given here) starts from Cauchy integrals for $f(z)$ and for the same function and its derivatives at $z = a$:

$$f(z) = \frac{1}{2\pi \imath} \oint_C \frac{dw f(w)}{w-z}, \quad f^{(n)}(a) = \frac{n!}{2\pi \imath} \oint_C \frac{dw f(w)}{(w-a)^{n+1}} \quad : n = 0, 1, \ldots$$

▷ For points z, a inside R and w on C we write,

$$\begin{aligned} \frac{1}{w-z} &= \frac{1}{(w-a)-(z-a)} = \frac{1}{w-a}\left[\frac{1}{1-\frac{z-a}{w-a}}\right] \\ &= \frac{1}{w-a}\left[1 + \frac{z-a}{w-a} + \left(\frac{z-a}{w-a}\right)^2 + \cdots + \left(\frac{z-a}{w-a}\right)^n + \cdots\right] \\ &= \frac{1}{w-a} + \frac{z-a}{(w-a)^2} + \frac{(z-a)^2}{(w-a)^3} + \cdots + \frac{(z-a)^n}{(w-a)^{n+1}} + \cdots . \end{aligned}$$

▷ Substituting these expressions, multiplied by $f(w)$, into the Cauchy integrals above yields the Taylor series as stated.

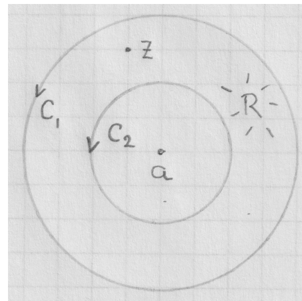
▷ A proof must demonstrate that this series converges. This can be done by (i) truncating the series at n, (ii) identifying the remainder term, and (iii) showing that the remainder vanishes as $n \to \infty$.

▷ The Taylor series looks no different than the power series of a real function. However, keep in mind that complex derivatives are subtle.

▷ In practice, the Taylor coefficients are rarely determined by contour integrals.

Laurent series:

Laurent's theorem generalizes Taylor's theorem to multiply connected regions of analyticity. The prototype is the annular region R between circles C_1 and C_2 centered at point a as shown. Note the sense of traversal of both C_1 and C_2.



If the function $f(z)$ is analytic in R and on C_1, C_2, then Laurent's theorem states that the following series with given a converges for all z in R:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-a)^n},$$

with coefficients extracted from integrals around the boundary circles:

$$a_n = \frac{1}{2\pi\imath} \oint_{C_1} dw \frac{f(w)}{(w-a)^{n+1}}, \quad a_{-n} = \frac{1}{2\pi\imath} \oint_{C_2} dw\, f(w)(w-a)^{n-1}.$$

The first and second sums constitute the *analytic* and *principal* parts.

▷ A proof (not given here) again starts from Cauchy integral for $f(z)$, which now involve the two boundary contours C_1 and C_2:

$$f(z) = \frac{1}{2\pi\imath} \oint_{C_1} \frac{dw f(w)}{w-z} - \frac{1}{2\pi\imath} \oint_{C_2} \frac{dw f(w)}{w-z},$$

▷ We expand $1/(w-z)$ for the first integral as done for the Taylor series:

$$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = \frac{1}{w-a}\left[\frac{1}{1-\frac{z-a}{w-a}}\right]$$
$$= \frac{1}{w-a} + \frac{z-a}{(w-a)^2} + \frac{(z-a)^2}{(w-a)^3} + \cdots + \frac{(z-a)^n}{(w-a)^{n+1}} + \cdots .$$

▷ We expand $-1/(w-z)$ for the second integral differently:

$$-\frac{1}{w-z} = \frac{1}{(z-a)-(w-a)} = \frac{1}{z-a}\left[\frac{1}{1-\frac{w-a}{z-a}}\right]$$
$$= \frac{1}{z-a}\left[1 + \frac{w-a}{z-a} + \left(\frac{w-a}{z-a}\right)^2 + \cdots + \left(\frac{w-a}{z-a}\right)^n + \cdots\right]$$
$$= \frac{1}{z-a} + \frac{w-a}{(z-a)^2} + \frac{(w-a)^2}{(z-a)^3} + \cdots + \frac{(w-a)^n}{(z-a)^{n+1}} + \cdots .$$

▷ Substituting the expansions into the Cauchy integral above yields the Laurent series as stated.

▷ Both circles C_1 and C_2 can be moved into the region R of analyticity with no change in the contour integrals. If we join them into a circle C we can simplify the expressions for the Laurent coefficients into

$$a_n = \frac{1}{2\pi\imath} \oint_C dw \frac{f(w)}{(w-a)^{n+1}} \quad : \; n = 0, \pm 1, \pm 2, \ldots$$

▷ Practical ways of determining the coefficients of a Laurent series are explored in [gex85] and [gex86].

▷ Laurent expansions at isolated singularities tell us the nature of that singularity:

- in a *removable singularity*, all a_{-n} vanish;
- in a *pole* of order n, then all $a_{-n'}$ with $n' > n$ vanish;
- in an *essential singularity*, infinitely many a_{-n} are nonzero.

▷ If $f(z)$ has a singularity at $z = \infty$, it can be analyzed as the singularity of $f(1/u)$ at $u = 0$.

▷ *Entire functions* are analytic everywhere except at $z = \infty$.

▷ *Meromorphic functions* are analytic for $|z| < \infty$ except for a finite number of poles.

▷ *Holomorphic functions* are, in essence, analytic functions (in a specified region of the complex plane).

Residues:

A single-valued function which has an isolated singularity at $z = a$ and is analytic inside and on a circle C around it can be expanded there into a convergent Laurent series:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n, \quad a_n = \frac{1}{2\pi\imath} \oint_C dz \, \frac{f(z)}{(z-a)^{n+1}}.$$

The coefficient a_{-1} has a special significance. It is called *residue*.

The residue is readily identified if we know the Laurent expansion. If the singularity is pole of order k, the following relation holds:

$$a_{-1} = \lim_{z\to a} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^k} \Big[(z-a)^k f(z) \Big].$$

The brute-force method, which works for all isolated singularities, uses

$$a_{-1} = \frac{1}{2\pi\imath} \oint_C dz f(z),$$

where the circle C must not surround or hit any other singularity.

Residue theorem:

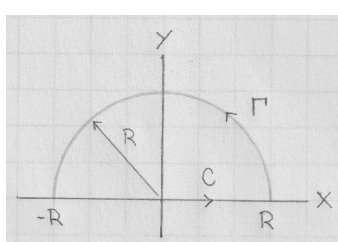
The contour integral of a single-valued function which is analytic inside and on the contour C except for isolated singularities depends only on the sum of residues of these singularities:

$$\oint_C dz \, f(z) = 2\pi\imath (a_{-1} + b_{-1} + c_{-1} + \ldots).$$

The residue theorem is a useful tool for the calculation of certain definite integrals. Here we consider the most common types of applications:

Application to rational functions:

▷ Consider contours in the form of semicircles of radius R in the upper-half complex plane:



▷ Consider rational functions $f(z)$ in which the maximum power of the denominator exceeds that of the numerator by two.

▷ Split the contour integral into two parts:

$$\underbrace{\oint_C dz\, f(z)}_{(a)} = \underbrace{\int_{-\infty}^{\infty} dx\, f(x)}_{(b)} + \underbrace{\lim_{R\to\infty} \int_\Gamma dz\, f(z)}_{(c)}.$$

▷ The contour integral (a) is determined by the residues of the isolated singularities inside the semicircle C.

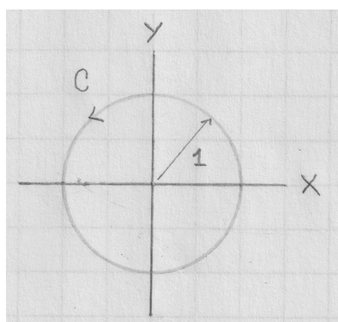
▷ The definite integral (b) along the real axis is the quantity of interest.

▷ The integral (c) along the semicircular path Γ must be shown to vanish in the limit $R \to \infty$:

$$|f(Re^{\imath\phi})| \leq \frac{M}{R^\alpha},\ \alpha > 1 \quad \Rightarrow \quad \left|\int_\Gamma dz\, f(z)\right| \leq \frac{M}{R^\alpha}\pi R \stackrel{R\to\infty}{\longrightarrow} 0.$$

Application to trigonometric functions:

▷ Consider a contour in the form of a unit circle centered at $z = 0$ in the complex plane:



▷ Consider rational functions $g(\cos\theta, \sin\theta)$.

▷ Transform the definite integral into a contour integral:

$$\int_0^{2\pi} d\theta\, g(\cos\theta, \sin\theta) = \int_C dz\, G(z).$$

▷ The transformation uses the ingredients,

$$z = e^{\imath\theta}, \quad d\theta = \imath z\, d\theta, \quad \cos\theta = \frac{1}{2}\big(z + z^{-1}\big), \quad \sin\theta = \frac{1}{2\imath}\big(z - z^{-1}\big).$$

▷ The contour integral is determined by the residues of all singularities inside C.

Application to Fourier transforms:

▷ Consider again contours in the form of semicircles of radius R in the upper-half complex plane (as in first application).

▷ Consider rational functions $f(z)$ in which the maximum power of the denominator exceeds that of the numerator by one.

▷ Split the contour integral into two parts:

$$\underbrace{\oint_C dz\, e^{\imath kz} f(z)}_{(a)} = \underbrace{\int_{-\infty}^{\infty} dx\, e^{\imath kx} f(x)}_{(b)} + \underbrace{\lim_{R\to\infty} \int_\Gamma dz\, e^{\imath kz} f(z)}_{(c)}.$$

▷ The contour integral (a) is determined by the residues of the isolated singularities inside the semicircle C.

▷ The definite integral (b) along the real axis is the Fourier transform $F(k)$ of the function $f(x)$.

▷ The integral (c) along the semicircular path Γ must be shown to vanish in the limit $R \to \infty$.

Premise: $|f(Re^{\imath\theta})| \leq \dfrac{M}{R^\alpha},\ \alpha > 0;\quad$ use $z = e^{\imath\theta},\ dz = \imath Re^{\imath\theta} d\theta$.

$$\begin{aligned}
\Rightarrow\quad \left|\int_\Gamma dz\, e^{\imath kz} f(z)\right| &= \left|\int_0^\pi d\theta\, e^{\imath kRe^{\imath\theta}} f(Re^{\imath\theta})\imath Re^{\imath\theta}\right| \\
&\leq \int_0^\pi d\theta \left|e^{\imath kR(\cos\theta + \imath\sin\theta)} f(Re^{\imath\theta})\imath Re^{\imath\theta}\right| \\
&= \int_0^\pi d\theta\, e^{-kR\sin\theta} |f(Re^{\imath\theta})| R \leq \frac{M}{R^{\alpha-1}} \int_0^\pi d\theta\, e^{-kR\sin\theta} \\
&= \frac{2M}{R^{\alpha-1}} \int_0^{\pi/2} d\theta\, e^{-kR\sin\theta} \leq \frac{2M}{R^{\alpha-1}} \int_0^{\pi/2} d\theta\, e^{-kR(2\theta/\pi)} \\
&= \frac{\pi M}{kR^\alpha}\left[1 - e^{-kR}\right] \stackrel{R\to\infty}{\longrightarrow} 0.
\end{aligned}$$