Complex Analysis I [gmd7-A]

Complex numbers:

The most common sets of numbers in order of increasing inclusiveness are denoted by symbols as follows:

$\mathbb{N}$: natural numbers,

$\mathbb{Z}$: integers,

$\mathbb{Q}$: rational numbers,

$\mathbb{R}$: real numbers,

$\mathbb{C}$: complex numbers.

Complex numbers are expressible as ordered pairs of real numbers. What can be accomplished with complex numbers is astonishing.

Complex number: $z \doteq x + \imath y, \quad x, y \in \mathbb{R}, \quad z \in \mathbb{C}.$

– Imaginary unit: $\imath \doteq \sqrt{-1}$,
– Real part: $x = \Re[z]$,
– Imaginary part: $y = \Im[z]$,
– Complex conjugate: $\bar{z} = x - \imath y$.

Extraction of real and imaginary parts:

$$\Re[z] = \frac{1}{2}(z + \bar{z}), \quad \Im[z] = \frac{1}{2\imath}(z - \bar{z}).$$

Rules of complex conjugation:

$$\bar{\bar{z}} = z, \quad \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2, \quad \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2.$$

Addition: $z_1 + z_2 = x_1 + x_2 + \imath(y_1 + y_2)$.

Multiplication: $z_1 z_2 = \underbrace{x_1 x_2 - y_1 y_2}_{\Re[z_1 z_2]} + \imath \underbrace{(x_1 y_2 + y_1 x_2)}_{\Im[z_1 z_2]}$.

Absolute value: $|z| \doteq \sqrt{z\bar{z}} = x^2 + y^2$.

Inverse: $z^{-1} = \dfrac{\bar{z}}{|z|^2}, \quad z \neq 0.$

Division: $\dfrac{z_1}{z_2} = \dfrac{z_1 \bar{z}_2}{z_2 \bar{z}_2} = \dfrac{z_1 \bar{z}_2}{|z_2|^2} = \underbrace{\dfrac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}}_{\Re[z_1/z_2]} + \imath \underbrace{\dfrac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2}}_{\Im[z_1/z_2]}, \quad z_2 \neq 0.$

Representation as ordered pair of real numbers: $z = (x, y)$.

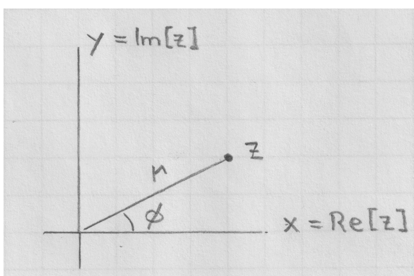
Functions of complex variables, $f(z)$, are, in general, complex themselves. The above rules and operations remain the same for functions.

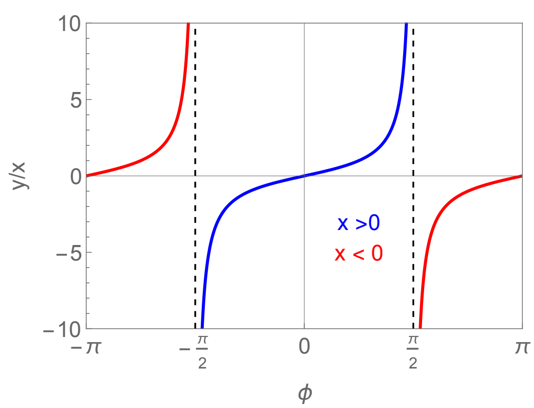
Polar representation: $z = \underbrace{r\cos\phi}_{x} + \imath \underbrace{r\sin\phi}_{y}$.

– Modulus: $r = |z| = \sqrt{x^2 + z^2}$,

– Argument:[1] $\phi = \arg z = \begin{cases} \arctan(y/x) & : x > 0, \\ \arctan(y/x) + \pi & : x < 0,\ y \geq 0, \\ \arctan(y/x) - \pi & : x < 0,\ y < 0, \\ (\pi/2)\mathrm{sgn}(y) & : x = 0. \end{cases}$

The principal range of the argument ϕ is $(-\pi, \pi)$ whereas that of $\arctan(y/x)$ is $(-\pi/2, \pi/2)$.





De Moivre theorem:

$$\begin{aligned} z_1 z_2 &= [r_1(\cos\phi_1 + \imath\sin\phi_1)][r_2(\cos\phi_2 + \imath\sin\phi_2)] \\ &= r_1 r_2[(\cos\phi_1\cos\phi_2 - \sin\phi_1\sin\phi_2) + \imath(\sin\phi_1\cos\phi_2 + \cos\phi_1\sin\phi_2)] \\ &= r_1 r_2[\cos(\phi_1 + \phi_2) + \imath\sin(\phi_1 + \phi_2)]. \end{aligned}$$

$$\Rightarrow \quad z^2 = [r(\cos\phi + \imath\sin\phi)]^2 = r^2[\cos(2\phi) + \imath\sin(2\phi)].$$

De Moivre's generalization:

$$z^n = [r(\cos\phi + \imath\sin\phi)]^n = r^n[\cos(n\phi) + \imath\sin(n\phi)].$$

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[1] The step function $\mathrm{sgn}(y)$ assumes the values $1, 0, -1$ for $y > 0, y = 0, y < 0$, respectively. Generalized functions (distributions) are discussed in [gmd3].

Roots of complex numbers: solutions of $w^n = z$ for given z.

$$w = z^{1/n} = r^{1/n}\left[\cos\left(\frac{\phi + 2\pi k}{n}\right) + \imath \sin\left(\frac{\phi + 2\pi k}{n}\right)\right] \quad : \ k = 0, 1, \ldots, n-1.$$

Euler's formula follows from examining series expansions:

$$\begin{aligned} e^{\imath\phi} &= \sum_{n=0}^{\infty} \frac{(\imath\phi)^n}{n!} = \sum_{k=0}^{\infty} \frac{\imath^{2k}\phi^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{\imath^{2k+1}\phi^{2k+1}}{(2k+1)!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k\phi^{2k}}{(2k)!} + \imath \sum_{k=0}^{\infty} \frac{(-1)^k\phi^{2k+1}}{(2k+1)!} = \cos\phi + \imath \sin\phi. \end{aligned}$$

Roots of unity: $z^n = 1 \quad \Rightarrow \ z = e^{2\pi\imath km/n} \quad : \ k = 0, 1, \ldots, n-1.$

Product revisited: $z_1 z_2 = r_1 r_2 e^{\imath(\phi_1 + \phi_2)}$,

- Modulus: $|z_1 z_2| = |z_1||z_2| = r_1 r_2$,
- Argument: $\arg(z_1 z_2) = \arg z_1 + \arg z_2 = \phi_1 + \phi_2$.

Polynomial equation: $a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0$.

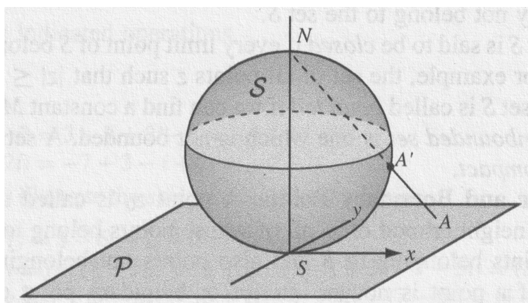
- Unique roots $z_1, \ldots, z_n$ are guaranteed to exist.
- Factorial form of polynomial: $a_0(z - z_1)(z - z_2)\cdots(z - z_n) = 0$.
- The roots are either real or complex.
- Some roots may be repeated.
- If all a_k are real then all complex roots occur as conjugate pairs.

Stereographic projection:

Consider a *Riemannian sphere* $\mathcal{S}$ of unit diameter touching the complex plane $\mathcal{P}$ at point S (south pole), which marks $z = 0$.

A straight line through point N (north pole) intersects the sphere at point A' and the plane at point A.

The stereographic projection is a one-on-one map between points on the sphere and points on the plane.



[image from Spiegel et al. 2009]

- The point S maps onto itself.
- Points on the equator of the sphere map onto points on the unit circle centered at $z = 0$ of the plane.
- The point N maps onto any point at infinity on the plane. In a sense, moving toward infinity in any direction on the plane means approaching the same point $z = \infty$.

Complex functions:

A single-valued complex function assigns one complex dependent variable w to a given complex independent variable z: $w = f(z)$.

Multiple-valued functions, familiar for real variables (e.g. the inverse sine) are a common occurrence in complex analysis.

Multiple-valued complex functions are regarded as a set of branches of single-valued functions, out of which one is selected as the *principal branch*.

There is more to complex functions than a mere extension from real to complex variables. The nature and attributes of functions are much more fully revealed in the complex plane. The German name for complex analysis is *Funktionentheorie* (theory of functions).

Elementary complex functions include the following:

- Polynomial functions: $P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$.
- Rational functions (ratio of polynomials): $\dfrac{P(z)}{Q(z)}$.
- Exponential function: $e^z \doteq \displaystyle\sum_{n=0}^{\infty} \frac{z^n}{n!}$.
- Trigonometric functions: $\sin z = \dfrac{1}{2\imath}\big(e^{\imath z} - e^{-\imath z}\big), \quad \cos z = \dfrac{1}{2}\big(e^{\imath z} + e^{-\imath z}\big)$, with $\tan z,\ \cot z,\ \sec z,\ \csc z$ constructed as in real functions.
- Hyperbolic functions: $\sinh z = \dfrac{1}{2}\big(e^{z} - e^{-z}\big), \quad \cosh z = \dfrac{1}{2}\big(e^{z} + e^{-z}\big)$, with $\tanh z,\ \coth z,\ \operatorname{sech} z,\ \operatorname{csch} z$ constructed as in real functions.
- Logarithmic function: $\ln z = \ln(re^{\imath\phi}) = \ln r + \imath(\phi + 2\pi k) \quad : \ k \in \mathbb{Z}$.
  Principal branch: $k = 0, \quad -\pi < \phi \leq \pi$.
- Inverse trigonometric functions (principal branch):

$$\arcsin z = \frac{1}{\imath}\ln\left(\imath z + \sqrt{1 - z^2}\right), \quad \arccos z = \frac{1}{\imath}\ln\left(z + \sqrt{z^2 - 1}\right),$$

$$\arctan z = \frac{1}{2\imath}\ln\left(\frac{1 + \imath z}{1 - \imath z}\right), \quad \operatorname{arccot} z = \frac{1}{2\imath}\ln\left(\frac{z + \imath}{z - \imath}\right).$$

- Inverse hyperbolic functions (principal branch):

$$\operatorname{arcsinh} z = \ln\left(z + \sqrt{z^2 + 1}\right), \quad \operatorname{arccosh} z = \ln\left(z + \sqrt{z^2 - 1}\right),$$

$$\operatorname{arctanh} z = \frac{1}{2}\ln\left(\frac{1 + z}{1 - z}\right), \quad \operatorname{arccoth} z = \frac{1}{2}\ln\left(\frac{z + 1}{z - 1}\right).$$

Remarks:

– The other branches of the inverse functions are obtained by adding $2\pi k$ to the variable of the logarithm.

– Note the close relationship between exponential, trigonometric, and hyperbolic functions for complex variables.

– Trigonometric functions for imaginary variables become hyperbolic functions and vice versa:

$$\sin(\imath z) = \imath \sinh z, \quad \cos(\imath z) = \cosh z, \quad \tan(\imath z) = \imath \tanh z,$$

$$\sinh(\imath z) = \imath \sin z, \quad \cosh(\imath z) = \cos z, \quad \tanh(\imath z) = \imath \tan z,$$

– The distinction between *algebraic* and *transcendental* functions $w(x)$ depends on whether or not $w(x)$ is a solution of the following equation for any polynomial functions $P_k(z)$ and positive integer n:

$$P_0(z)w^n + P_1(z)w^n + \cdots + P_{n-1}(z) + P_n = 0,$$

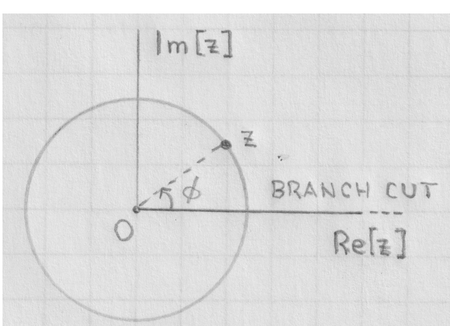
Branch cut and Riemann surface:

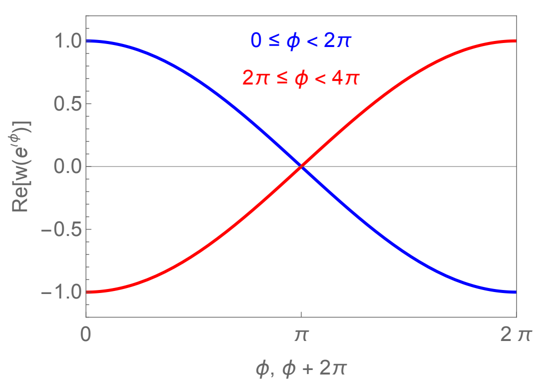
Multiple-valued functions can be made single-valued via two distinct strategies, here described for the double-valued function $w(z) = \sqrt{z}$.

$$z = re^{\imath\phi} \quad \Rightarrow \quad w(z) = \sqrt{r}e^{\imath(\phi+2\pi k)/2} \quad : \quad k = 0, 1.$$

Increasing the argument ϕ at fixed modulus $r = 1$ reveals the two-valued nature of the function:

$$w\left(e^{\imath(\phi+2\pi)}\right) = -w\left(e^{\imath\phi}\right), \quad w\left(e^{\imath(\phi+4\pi)}\right) = w\left(e^{\imath\phi}\right).$$





– *Branch cut*: [blue line only for $0 \leq \phi < 2\pi$]
  The two branches are connected by the branch cut $\phi = 0$ (positive real axis). Points at the end of a branch cut are named *branch points*. here O is one branch points, the other is at infinity. If we restrict any variation of z not to cross the branch cut and to avoid branch points, we have a single-valued function $w(z) = \sqrt{z}$.

– *Riemann surface*: [blue line for $0 \leq \phi < 2\pi$, red line for $2\pi \leq \phi < 4\pi$]
  The two branches of the function cover the complex plane in two sheets. The two branches are connected along the branch cut. Any variation of z that crosses the cut moves from one branch to the other branch. The function $w(z) = \sqrt{z}$ is single-valued on the two-sheet Riemann surface.

Limits and continuity:

– The limit $\lim_{z \to z_0} f(z)$ of a complex function may or may not exist.

– If the limit exists it is unique.

– Limit at infinity: $\lim_{z \to \infty} f(z) = \lim_{w \to 0} f(1/w)$.

– Limits of sums and products of functions are sums or products of limits.

– If $\lim_{z \to z_0} f(z) = f(z_0)$ then the function $f(z)$ is continuous at $z = z_0$.

– Isolated discontinuities are removable.

– A function is continuous in a region if it is continuous at all points of the region

– Composite functions (functions of functions) constructed from continuous functions are continuous.

– The real and imaginary parts of a continuous function are continuous.

– A function which is continuous in a closed and bounded region is called *uniformly continuous*.

Complex derivatives:

Complex variable: $z = x + \imath y$.

Complex function:[2] $f(z) = u(x,y) + \imath\, v(x,y) = \Re[f(z)] + \imath\, \Im[f(z)]$.

Complex derivative defined as limit of quotient:

$$f'(z) = \frac{df}{dz} \doteq \lim_{dz\to 0} \frac{f(z+dz)-f(z)}{dz}.$$

This definition only makes sense if it produces a unique result.

Two alternative ways of carrying out derivatives:

– Use chain rule for complex variable:

$$\frac{\partial f}{\partial x} = \frac{df}{dz}\underbrace{\frac{\partial z}{\partial x}}_{1} = \frac{df}{dz}, \quad \frac{\partial f}{\partial y} = \frac{df}{dz}\underbrace{\frac{\partial z}{\partial y}}_{\imath} = \imath\,\frac{df}{dz}.$$

– Use real and imaginary parts of complex function:

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + \imath\frac{\partial v}{\partial x}, \quad \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + \imath\frac{\partial v}{\partial y}.$$

This yields two distinct ways of expressing the complex derivative:

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + \imath\frac{\partial v}{\partial x}, \quad \imath\,\frac{df}{dz} = \frac{\partial u}{\partial y} + \imath\frac{\partial v}{\partial y}.$$

$$\Rightarrow\ \frac{df}{dz} = \frac{\partial u}{\partial x} + \imath\,\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - \imath\,\frac{\partial u}{\partial y}.$$

The complex derivative does not exist for just any complex function whose real and imaginary parts have well-defined partial derivatives with respect to the real and imaginary parts of the complex variable.

A complex function is *analytic* (complex differentiable) in a region of z if its real and imaginary parts satisfy the *Cauchy-Riemann conditions* :

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

If these conditions check out we can carry out the derivative as follows:

$$\frac{df}{dz} = \frac{\partial u}{\partial x} - \imath\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + \imath\frac{\partial v}{\partial x} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \imath\frac{\partial}{\partial y}\right)(u + \imath v).$$

[2]The real functions $u(x,y)$ and $v(x,y)$ are subject to certain conditions (discussed below) if they constitute a complex function expressible as $f(z)$. General complex functions are expressible in the form $f(z,\bar{z})$.

In like manner, the complex derivatives with respect to $\bar{z}$ yields

$$\frac{df}{d\bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \imath\frac{\partial}{\partial y}\right)(u + \imath v).$$

It follows that for any analytic function, i.e. for any function that satisfies the Cauchy-Riemann conditions, we have

$$\frac{df}{d\bar{z}} = \frac{1}{2}\left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + \imath\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] = 0.$$

The implication is that a complex function $f(x, y)$ with $x = (z + \bar{z})/2$ and $y = (z - \bar{z})/2\imath$ is analytic if it depends on z but not on $\bar{z}$.

General complex function: $A(x, y) = B(z, \bar{z})$ with $z = x + \imath y,\ \bar{z} = x - \imath y$.

Analytic function: $A(x, y) = B(z)$.

Second derivatives applied to Cauchy-Riemann conditions:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}, \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y}, \quad \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2}, \quad \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2}.$$

In consequence, the real and imaginary parts of an analytic function satisfy the Laplace equation separately.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Solutions $u(x, y)$ and $v(x, y)$ of the Laplace equation are named *harmonic functions*.

The functions $u(x, y),\ v(x, y)$ which are the real and imaginary parts of an analytic function $f(z) = u + \imath v$ are named *conjugate functions*.

Not every pair functions $u(x, y),\ v(x, y)$ make a conjugate pair, even if they (independently) are harmonic functions. The Cauchy-Riemann conditions are more stringent.

Given a harmonic function, a conjugate harmonic function to it can be constructed such that together they constitute an analytic function.

The rules for differentiation applied to analytic functions are identical to the familiar rules for real functions of real variables. If anything, some of the derivatives become more transparent.

L'Hospital's rule: If $f(z),\ g(z)$ are analytic in some region and vanish at the same point z_0: $f(z_0) = g(z_0) = 0$, then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)} \quad \text{if } g'(z_0) \neq 0.$$

Points in the complex plane at which complex functions are not analytic are called *singular points*, of which different types must be distinguished.

- *Isolated singularities* can be surrounded by a circle of nonzero radius which contains no other singularity.
- *Poles* of order n are isolated singularities with the property that

  $$\lim_{z \to z_0} (z - z_0)^n f(z) \neq 0.$$

  For *simple poles* we have $n = 1$.
- *Branch points* are non-isolated singular points present in multiple-valued functions as encountered earlier.
- *Removable singularities* are isolated singularities, (e.g. at z_0) for which $\lim_{z \to z_0} f(z)$ exists. The replacement of $f(z_0)$ by the limit removes the singulariy and makes the function analytic at z_0.
- *Essential singularities* are isolated singularities which are not removable and are not poles.
- *Singularities at infinity* such as in $f(z)$ at $z = \infty$ are classified as singularities of $f(1/w)$ at $w = 0$.

Differential operators:

The differential operators familiar from vector analysis in $\mathbb{R}^3$ [gmd1] can be adapted to complex functions, $A(x,y) = P(x,y) + \imath Q(x,y) = B(z,\bar{z})$.

– Gradient:

$$\nabla A \doteq \left(\frac{\partial}{\partial x} + \imath \frac{\partial}{\partial y}\right) A(x,y) = 2\,\frac{\partial}{\partial \bar{z}}\,B(z,\bar{z}).$$

We have used the chain rule:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}, \quad \frac{\partial}{\partial y} = \imath\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right) \quad \Rightarrow \quad \frac{\partial}{\partial x} + \imath\,\frac{\partial}{\partial y} = 2\frac{\partial}{\partial \bar{z}}.$$

– Divergence:

$$\begin{aligned}\nabla \cdot A &\doteq \frac{\partial}{\partial x}\,\Re\{A(x,y)\} + \frac{\partial}{\partial y}\,\Im\{A(x,y)\} \\ &= \Re\left\{\left(\frac{\partial}{\partial x} - \imath\,\frac{\partial}{\partial y}\right) A(x,y)\right\} = 2\Re\left\{\frac{\partial}{\partial z}\,B(z,\bar{z})\right\}.\end{aligned}$$

We have used the chain rule again: $\dfrac{\partial}{\partial x} - \imath\,\dfrac{\partial}{\partial y} = 2\dfrac{\partial}{\partial z}$.

– Curl:

$$\begin{aligned}\nabla \times A &\doteq \frac{\partial}{\partial x}\,\Im\{A(x,y)\} - \frac{\partial}{\partial y}\,\Re\{A(x,y)\} \\ &= \Im\left\{\left(\frac{\partial}{\partial x} - \imath\,\frac{\partial}{\partial y}\right) A(x,y)\right\} = 2\Im\left\{\frac{\partial}{\partial z}\,B(z,\bar{z})\right\}.\end{aligned}$$

– Laplacian:

$$\begin{aligned}\nabla^2 A &\doteq \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) A(x,y) \\ &= \Re\left\{\left(\frac{\partial}{\partial x} - \imath\,\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + \imath\,\frac{\partial}{\partial y}\right)\right\} A(x,y) = 4\,\frac{\partial^2}{\partial z \partial \bar{z}}\,B(z,\bar{z}).\end{aligned}$$

We have used:

$$\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial z^2} + 2\frac{\partial^2}{\partial z \partial \bar{z}} + \frac{\partial^2}{\partial \bar{z}^2}, \quad \frac{\partial^2}{\partial y^2} = -\frac{\partial^2}{\partial z^2} + 2\frac{\partial^2}{\partial z \partial \bar{z}} - \frac{\partial^2}{\partial \bar{z}^2}.$$

If $A(x,y) = B(z)$ (analytic function), the gradient and the Laplacian vanish. Then the real functions $P(x,y)$ and $Q(x,y)$ are harmonic.

Orthogonal families of curves:

Analytic function: $f(z) = g(x, y) + \imath h(x, y)$.

Conjugate functions: $g(x, y), \quad h(x, y)$.

Cauchy-Riemann conditions: $\dfrac{\partial g}{\partial x} = \dfrac{\partial h}{\partial y}, \quad \dfrac{\partial g}{\partial y} = -\dfrac{\partial h}{\partial x}$.

The equations $g(x, y) = \alpha$, $h(x, y) = \beta$ represent two one-parameter families of curves in the xy-plane.

The product of slopes of two curves intersecting at right angle is -1. This can be shown to be the case for any two curves $g(x, y) = \alpha$, $h(x, y) = \beta$.

▷ Use differential of $g(x, y) = $ const.:

$$\Rightarrow \; dg = \frac{\partial g}{\partial x}\, dx + \frac{\partial g}{\partial y}\, dy = 0 \quad \Rightarrow \left(\frac{dy}{dx}\right)_g = -\frac{\partial g/\partial x}{\partial g/\partial y}.$$

▷ Use differential of $h(x, y) = $ const.:

$$\Rightarrow \; dh = \frac{\partial h}{\partial x}\, dx + \frac{\partial h}{\partial y}\, dy = 0 \quad \Rightarrow \left(\frac{dy}{dx}\right)_h = -\frac{\partial h/\partial x}{\partial h/\partial y}.$$

▷ Use the Cauchy-Riemann conditions:

$$\Rightarrow \; \frac{\partial h/\partial x}{\partial h/\partial y} = -\frac{\partial g/\partial y}{\partial g/\partial x}.$$

▷ Relate the slopes of the two curves:

$$\Rightarrow \; \left(\frac{dy}{dx}\right)_h = -\left(\frac{dy}{dx}\right)_g^{-1} \quad \Rightarrow \; \left(\frac{dy}{dx}\right)_h \left(\frac{dy}{dx}\right)_g = -1.$$