Tensor Analysis II [gmd5-B]

Tensor operations:

We are familiar with matrix operations [gmd6] and vector operations [gmd1]. We know that matrices or vectors are tensors under specific conditions. Here we introduce the most basic operations involving tensors.

- *Sum*: A linear combination of tensors of the same rank and order is again a tensor of that rank and order.
- *Outer product*:[1] If $\mathbf{A}$ is a tensor of order (p, q) and $\mathbf{B}$ a tensor of order (r, s), then its outer product is a tensor of order $(p+r, q+s)$.

  $$\mathbf{A} \otimes \mathbf{B} = \mathbf{B} \otimes \mathbf{A} = \mathbf{C}. \tag{1}$$

  Example: $A_i^{jl} B_{mn}^{op} = C_{imn}^{jlop}$.

  Tensors cannot, in general, be expressed as outer products of lower-order tensors.
- *Contraction*: If $\mathbf{A}$ is a tensor of order (p, q), then equating one superscript index and one subscript index implies a summation over them, which reduces the the order to $(p-1, q-1)$.

  Example: $C_{imn}^{jlon} = D_{im}^{jlo}$.

  The contraction of a rank-2 tensor (matrix) produces its (scalar) trace:

  $$\mathrm{Tr}[\mathbf{A}] = A_i^i. \tag{2}$$
- *Inner product*: An inner product of two tensors is its outer product followed by one or several contractions.

  Example: $A_i^{jl} B_{mn}^{op} = C_{imn}^{jlop} \stackrel{n=p}{\longrightarrow} C_{imn}^{jlon} = D_{im}^{jlo}$.

  Not all contractions start with an outer product.

  Inner product of two rank-1 tensors produces a scalar (rank-0 tensor):

  $$\mathbf{A} \cdot \mathbf{B} = C, \qquad A^i B_i = C. \tag{3}$$

  The outer product of two rank-2 tensors contracted once amounts to a matrix product and contracted twice yields the trace of the product matrix:

  $$\mathbf{AB} = A_i^j B_j^k = C_i^k = \mathbf{C}, \qquad \mathrm{Tr}[\mathbf{C}] = C_i^i. \tag{4}$$

[1] Outer products applied to tensors are also named *tensor* products. Outer products of quantities that are not tensors are discussed elsewhere. The outer product of two vectors (rank-1 tensors) of equal dimension is also named *dyadic* product.

Quotient rule:

If the inner product of **B** with a tensor **A** can be carried out and shown to be a tensor **C**, then **B** must be a tensor too.

Applications:

#1 If $T_i V^i \doteq E$ is an invariant for all contravariant vectors V^i, then the array T_i is a covariant vector.

$$\bar{V}^i = V^j \frac{\partial \bar{x}^i}{\partial x^j}, \quad \bar{E} \doteq \bar{T}_i \bar{V}^i = T_i V^i \doteq E \quad \Rightarrow \bar{T}_i \left(V^j \frac{\partial \bar{x}^i}{\partial x^j} \right) = T_i V^i = T_j V^j$$

$$\Rightarrow \quad \left(T_j - \bar{T}_i \frac{\partial \bar{x}^i}{\partial x^j} \right) V^j = 0 \text{ for all } V^j \quad \Rightarrow \; T_j = \bar{T}_i \frac{\partial \bar{x}^i}{\partial x^j}.$$

#2 If $U_j = T_{ij} V^i$ is a covariant vector for all contravariant vectors V^i, then the array T_{ij} is a covariant rank-2 tensor [gex118].

#3 If $T_{ij} U^i V^j \doteq E$ is an invariant for all contravariant vectors U^i and V^j, then the array T_{ij} is a covariant rank-2 tensor [gex118].

#4 If the array T_{ij} is symmetric ($T_{ij} = T_{ji}$) and $T_{ij} V^i V^j$ is invariant for all contravariant vectors V^i, then T_{ij} is a covariant rank-2 tensor [gex119].

Arrays of elements – vectors, matrices, tensors:

Not every matrix or vector is a tensor, not every tensor is a matrix or a vector, but all of them are arrays or lists of elements.

Vectors and matrices qualify as rank-1 and rank-2 tensors, respectively, if they satisfy specific conditions under specific coordinate transformations. In this section, we only consider the identity transformation. All vectors and matrices are tensors for this case.

Array (named List in Mathematica) is the most neutral term for a structured collection of elements (numbers or functions). Vectors, matrices, and tensors are arrays with specific structures. The structure of arrays is henceforth indicated by curly braces { }.

Consider vectors in 3D space: $\mathbf{V} = \{v_1, v_2, v_3\}$, $\quad \mathbf{W} = \{w_1, w_2, w_3\}$.

- **V** and **W** qualify as tensors:
  - ▷ Mathematica: TensorRank: 1,
  - ▷ Mathematica: TensorDimensions: {3}
- The dot product is an invariant:
  - ▷ $\mathbf{V} \cdot \mathbf{W} = v_1 w_1 + v_2 w_2 + v_3 w_3$.
  - ▷ Mathematica: Dot[V,W].
- The cross product is a vector and qualifies as a rank-1 tensor:
  - ▷ $\mathbf{V} \times \mathbf{W} = \{v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1\}$.
  - ▷ Mathematica: Cross[V,W].
- The tensor product is a matrix and qualifies as a rank-2 tensor:
  - ▷ $\mathbf{U} = \mathbf{V} \otimes \mathbf{W}$
    $= \{\{v_1 w_1, v_1 w_2, v_1 w_3\}, \{v_2 w_1, v_2 w_2, v_2 w_3\}, \{v_3 w_1, v_3 w_2, v_3 w_3\}\}$.
  - ▷ Mathematica: U=TensorProduct[V,W].
  - ▷ TensorRank[U]: 2, TensorDimensions[U] $\{3, 3\}$.
  - ▷ MatrixForm[U]: $\begin{pmatrix} v_1 w_1 & v_1 w_2 & v_1 w_3 \\ v_2 w_1 & v_2 w_2 & v_2 w_3 \\ v_3 w_1 & v_3 w_2 & v_3 w_3 \end{pmatrix}$.
- The contraction yields the trace, which is the dot product.
  - ▷ Mathematica: Tr[U]: $v_1 w_1 + v_2 w_2 + v_3 w_3$,
  - ▷ TensorContract[U,{1,2}]: $v_1 w_1 + v_2 w_2 + v_3 w_3$.