Bessel Functions [gmd4F]

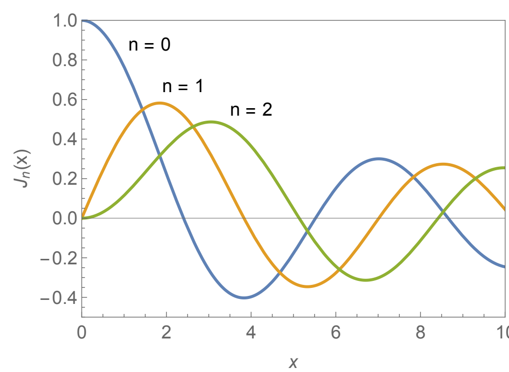
Solutions of partial differential equations for problems with cylindrical symmetry are often expressible as Bessel functions.

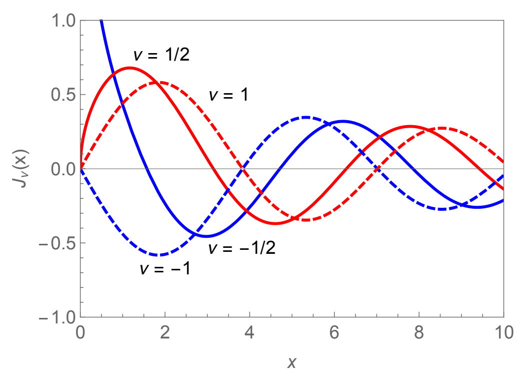
Bessel equation: $z^2 R''(z) + zR'(z) + (z^2 - \nu^2)R(z) = 0$.

Bessel functions of the first kind: $\mathrm{J}_\nu(z) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\Gamma(s+\nu+1)} \left(\frac{z}{2}\right)^{2s+\nu}$.

For noninteger order ν, the function $\mathrm{J}_\nu(z)$ and $\mathrm{J}_{-\nu}(z)$ are linearly independent. Their linear dependence for integer ν is manifest in the relation,[1]

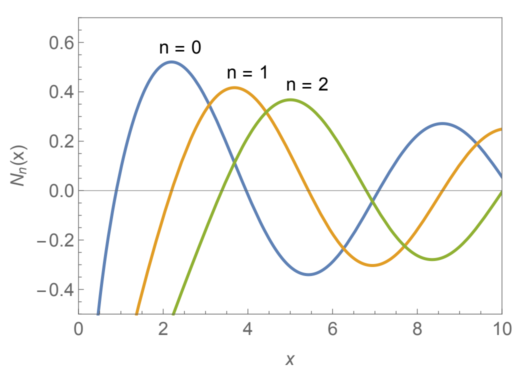
$$\mathrm{J}_{-\nu}(z) = (-1)^\nu \mathrm{J}_\nu(z) \quad : \ \nu \in \mathbb{Z}.$$





A solution which remains linearly independent of $\mathrm{J}_\nu(x)$ is the *Neumann function* (Bessel function of the second kind) constructed as follows:

$$\mathrm{N}_\nu(z) \doteq \frac{\mathrm{J}_\nu(z)\cos(\nu\pi) - \mathrm{J}_{-\nu}(z)}{\sin(\nu\pi)}.$$



The divergence at $x = 0$ is logarithmic in nature.

[1]This relation emerges on account of the fact that the function $\Gamma(x)$ diverges for non-positive integers.

Modified Bessel functions:

A 90° rotation in the complex plane converts the Bessel functions into modified versions such as the folowig:

$$\mathrm{I}_\nu(z) = \imath^{-\nu}\mathrm{J}_\nu(\imath z).$$

They are solutions of the modified Bessel equation,

$$z^2R''(z) + zR'(z) - (z^2+\nu^2)R(z) = 0,$$

and have quite different properties for real z.

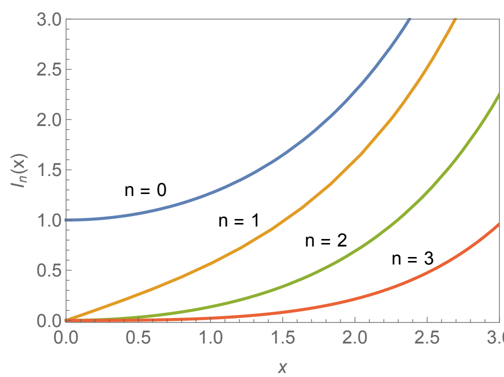
Modified Bessel function of the first kind: $\mathrm{I}_\nu(z) = \sum_{s=0}^{\infty} \frac{1}{s!\Gamma(s+\nu+1)}\left(\frac{z}{2}\right)^{2s+\nu}$.

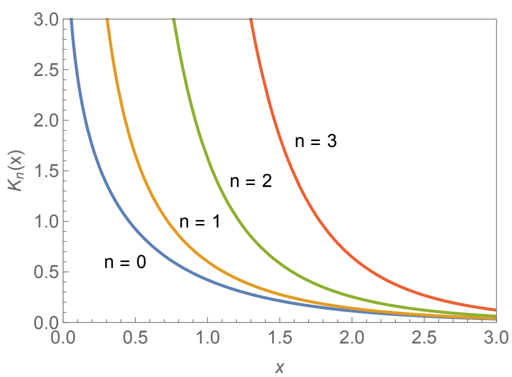
For noninteger ν, the functions $\mathrm{I}_\nu(z)$ and $\mathrm{I}_{-\nu}(z)$ are again linearly independent, whereas for integer ν we have,

$$\mathrm{I}_{-\nu}(z) = \mathrm{I}_\nu(z) \quad : \ \nu \in \mathbb{Z}.$$

A solution which remains linearly independent of $\mathrm{I}_\nu(z)$ for integer ν is the *MacDonald function* (modified Bessel function of the second kind):

$$\mathrm{K}_\nu(z) \doteq \frac{\pi}{2}\frac{\mathrm{I}_{-\nu}(z) - \mathrm{I}_\nu(z)}{\sin(\nu\pi)}.$$





Useful relations:

- ▷ $\dfrac{d}{dz}\big[z^{\nu}\mathrm{J}_{\nu}(z)\big] = z^{\nu}\mathrm{J}_{\nu-1}(z), \quad \dfrac{d}{dz}\big[z^{-\nu}\mathrm{J}_{\nu}(z)\big] = -z^{-\nu}\mathrm{J}_{\nu+1}(z).$
- ▷ $\mathrm{J}'_{\nu}(z) = \mathrm{J}_{\nu-1}(z) - \dfrac{\nu}{z}\mathrm{J}_{\nu}(z), \quad \mathrm{J}'_{\nu}(z) = -\mathrm{J}_{\nu+1}(z) + \dfrac{\nu}{z}\mathrm{J}_{\nu}(z).$
- ▷ $\mathrm{J}'_{\nu}(z) = \dfrac{1}{2}\big[\mathrm{J}_{\nu-1}(z) - \mathrm{J}_{\nu+1}(z)\big].$
- ▷ $\mathrm{J}_{\nu}(z) = \dfrac{1}{2}\,\nu z\big[\mathrm{J}_{\nu-1}(z) + \mathrm{J}_{\nu+1}(z)\big].$
- ▷ $\mathrm{J}_0(z) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} d\theta\, e^{\imath z\cos\theta}.$
- ▷ $|z| \ll 1: \quad \mathrm{J}_{\nu}(z) \rightsquigarrow \dfrac{(z/2)^{\nu}}{\Gamma(\nu+1)} \quad : \ \nu \neq 0, -1, -2, \ldots,$
  power-law rise from zero or power-law divergence.
- ▷ $|z| \gg 1: \quad \mathrm{J}_{\nu}(z) \rightsquigarrow \sqrt{\dfrac{2}{\pi z}}\cos\Big(z + \big(\nu + \tfrac{1}{2}\big)\,\dfrac{\pi}{2}\Big),$
  attenuated oscillation with ν-dependent phase shift.