Spherical Harmonics [gmd4E]

Any function $g(\theta,\phi)$ defined on the unit sphere and expressed by polar angle θ and azimuthal angle ϕ can be expanded as a series of spherical harmonics:

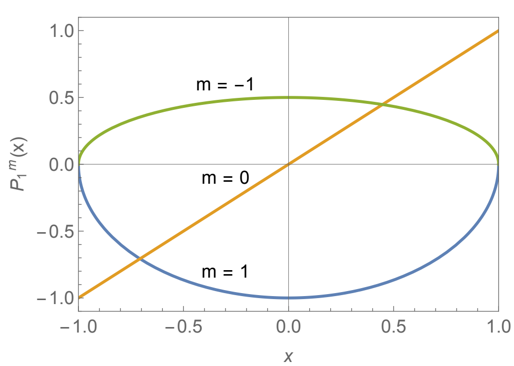
$$g(\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}C_{lm}Y_{lm}(\theta,\phi),\quad C_{lm}=\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\sin\theta\,Y^*_{lm}(\theta,\phi)g(\theta,\phi).$$

Spherical harmonics are a complete set of orthonormal functions, composed of associated Legendre functions for θ and harmonic oscillations for ϕ:

$$Y_{lm}(\theta,\phi)\doteq\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{l+m)!}}P_l^m(\cos\theta)e^{\imath m\phi}.$$

Associated Legendre functions:

- Rodrigues' generator: $P_l^m(u)=\dfrac{(-1)^m}{2^l l!}(1-u^2)^{m/2}\dfrac{d^{l+m}}{du^{l+m}}(u^2-1)^l.$
- Relation to polynomials: $P_l^m(u)=(-1)^m(1-u^2)^{m/2}\dfrac{d^m}{du^m}P_l(u).$
- Orthogonality: $\displaystyle\int_{-1}^{+1}duP_{l'}^m(u)P_l^m(u)=\frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\,\delta_{l'l}.$
- Relation: $P_l^{-m}(u)=(-1)^m\dfrac{(l-m)!}{(l+m)!}P_l^m(u).$
- Case $m=0$: $P_l^0(x)=P_l(x)$.
- Case $l=1$: $P_1^1(x)=-\sqrt{1-x^2},\quad P_1^0(x)=x,\quad P_1^{-1}(x)=\frac{1}{2}\sqrt{1-x^2}.$



- Case $l=2$: $P_2^2(x)=3(1-x^2),\quad P_2^1(x)=-3x\sqrt{1-x^2},$
  $P_2^0(x)=\frac{1}{2}(3x^2-1),\quad P_2^{-1}(x)=\frac{1}{2}x\sqrt{1-x^2},$
  $P_2^{-2}(x)=\frac{1}{8}(1-x^2).$

Attributes of spherical harmonics:

- Complex conjugate function: $Y^*_{lm}(\theta,\phi) = (-1)^m Y_{l,-m}(\theta,\phi)$.
- Orthonormality: $\displaystyle\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta\, Y^*_{l'm'}(\theta,\phi) Y_{lm}(\theta,\phi) = \delta_{l'l}\delta_{m'm}$.
- Completeness: $\displaystyle\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y^*_{lm}(\theta',\phi') Y_{lm}(\theta,\phi) = \delta(\phi-\phi')\delta(\cos\theta - \cos\theta')$.
- Case $l=0$: $Y_{00} = \dfrac{1}{\sqrt{4\pi}}$.
- Case $l=1$: $Y_{11} = -\sqrt{\dfrac{3}{8\pi}}\sin\theta e^{\imath\phi}$, $Y_{10} = \sqrt{\dfrac{3}{4\pi}}\cos\theta$, $Y_{1,-1} = \sqrt{\dfrac{3}{8\pi}}\sin\theta e^{-\imath\phi}$.
- Case $l=2$: $Y_{22} = \sqrt{\dfrac{15}{32\pi}}\sin^2\theta e^{2\imath\phi}, \quad Y_{21} = -\sqrt{\dfrac{15}{8\pi}}\sin\theta\cos\theta e^{\imath\phi}$,
  $Y_{20} = \sqrt{\dfrac{5}{4\pi}}\left(\tfrac{3}{2}\cos^2\theta - \tfrac{1}{2}\right)$,
  $Y_{2,-1} = \sqrt{\dfrac{15}{8\pi}}\sin\theta\cos\theta e^{-\imath\phi}, \quad Y_{2,-2} = \sqrt{\dfrac{15}{32\pi}}\sin^2\theta e^{-2\imath\phi}$.

Addition theorem for spherical harmonics:

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y^*_{lm}(\theta',\phi') Y_{lm}(\theta,\phi),$$

where γ is the angle between position vector $\mathbf{x}$ with angular coordinates θ, ϕ and position vector $\mathbf{x}'$ with angular coordinates θ', ϕ'. The angles are thus related as follows: $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$.

Sum rule emerging in the limit $\gamma \to 0$:

$$\sum_{m=-l}^{l} |Y_{lm}(\theta,\phi)|^2 = \frac{2l+1}{4\pi}.$$

Application to electostatic potential of point charge:

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y^*_{lm}(\theta',\phi') Y_{lm}(\theta,\phi),$$

where $r_<$ and $r_>$ are the magnitudes of the shorter and longer vectors $\mathbf{x}$ and $\mathbf{x}'$, respectively.