Legendre Polynomials and Functions [gmd4D]

Legendre polynomials:

Regular solutions of Legendre equation form a complete set of orthogonal polynomial functions $P_l(u)$ with range $-1 \leq u \leq 1$.

Legendre equation: $\frac{d}{du}\left[(1-u^2)\frac{dP}{du}\right] + l(l+1)P = 0.$

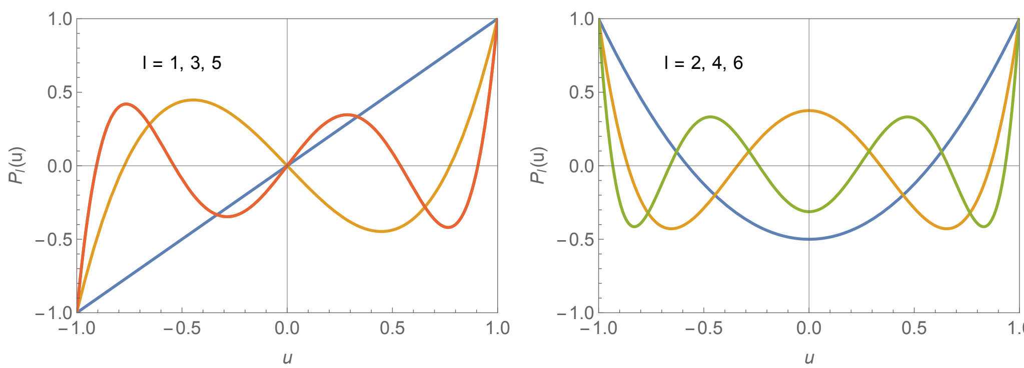
Ansatz: $P(u) = \sum_{n=0}^{\infty} C_n u^n.$

Substitution into Legendre equation yields recursion relations for the C_n for even and odd polynomials with built-in termination:

$$C_0 \doteq 1, \quad C_1 \doteq u, \quad C_{n+2} = C_n \frac{n(n+1) - l(l+1)}{(n+2)(n+1)}.$$

Legendre polynomials for $l \leq 4$:

$$P_0(u) = 1, \quad P_1(u) = u, \quad P_2(u) = \frac{1}{2}(3u^2 - 1),$$
$$P_3(u) = \frac{1}{2}(5u^3 - 3u), \quad P_4(u) = \frac{1}{8}(35u^4 - 30u^2 + 3).$$



Normalization: $P_l(1) = 1.$

Orthogonality: $\int_{-1}^{+1} du\, P_l(u) P_{l'}(u) = \frac{2\delta_{ll'}}{2l+1}.$

Rodrigues' generator: $P_l(u) = \frac{1}{2^l\, l!}\frac{d^l}{du^l}(u^2 - 1)^l.$

Recurrence relations:

$$
\begin{aligned}
&(l+1)P_{l+1}(u) - (2l+1)uP_l(u) + lP_{l-1}(u) = 0, \\
&\frac{d}{du}\,P_{l+1}(u) - u\,\frac{d}{du}\,P_l(u) - (l+1)P_l(u) = 0, \\
&(u^2-1)\frac{d}{du}\,P_l(u) - luP_l(u) + lP_{l-1}(u) = 0.
\end{aligned}
$$

Orthogonal expansion: $f(u) = \sum_{l=0}^{\infty} a_l P_l(u).$

Expansion coefficients: $a_l = \frac{2l+1}{2}\int_{-1}^{+1} du\, P_l(u) f(u).$

Even functions: $f(-u) = f(u) \quad \Rightarrow\ a_l = 0$ for odd l.

Odd functions: $f(-u) = -f(u) \quad \Rightarrow\ a_l = 0$ for even l.

Generating function: $g(u,s) \doteq \sum_{l=0}^{\infty} s^l P_l(u) \quad : \ |s| < 1 \qquad$ (definition).

Generation of Legendre polynomials: $P_l(u) = \frac{1}{l!}\,\frac{\partial^l g}{\partial u^l}\bigg|_{s=0} \ : \ l = 0, 1, 2, \ldots$

Legendre equation (ODE) adapted for generating function becomes PDE:

$$
\frac{\partial}{\partial u}\left[\big(1-u^2\big)\frac{\partial g}{\partial u}\right] + s\,\frac{\partial^2}{\partial s^2}\big(sg\big) = 0. \tag{1}
$$

Initial conditions and boundary conditions:

$$
g(u,0) = P_0(u) = 1, \quad \frac{\partial}{\partial s} g(u,s)\Big|_{s=0} = P_1(u) = u, \tag{2}
$$

$$
g(1,s) = \sum_{l=0}^{\infty} s^l = \frac{1}{1-s}, \quad g(-1,s) = \sum_{l=0}^{\infty} (-1)^l s^l = \frac{1}{1+s}. \tag{3}
$$

Solution of (1) with conditions (2) and (3):

$$
g(u,s) = \frac{1}{\sqrt{1-2us+s^2}} = \sum_{l=0}^{\infty} s^l P_l(u). \tag{4}
$$

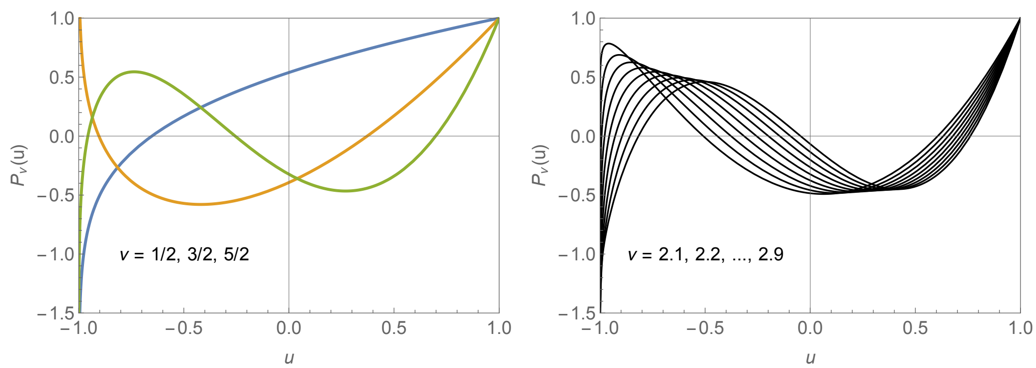
Legendre functions:

For some applications, it is necessary to consider solutions of the Legendre equation for non-integer l, named ν henceforth.

Such solutions, named Legendre functions, are special cases of hypergeometric functions:

$$P_\nu(u) = {}_2F_1\left(-\nu, \nu+1; 1; \frac{1-u}{2}\right).$$

Unlike Legendre polynomials, these functions are singular at $u = -1$.



The leading singularity for $\nu = \frac{1}{2}$ is $P_{1/2}(u) \sim \ln(1+u)$.

The polynomial $P_l(u)$ has exactly l zeros. The number of zeros in the functions $P_\nu(u)$ increases with ν.