Elliptic Integrals and Elliptic Functions [gmd4C]

The ubiquity and importance of elliptic integrals is explained by a theorem here stated without proof:

If $R(x, y)$ is a rational function of x and y, and $P(x)$ is a polynomial of degree four or less, then the indefinite integral, $\int dx\, R(x, \sqrt{P(x)})$, can be expressed as elliptic integrals.

We distinguish between *incomplete* and *complete* elliptic integrals. The former are indefinite integrals and the latter definite integrals.

Incomplete elliptic integrals:

Definitions of incomplete elliptic integrals use different arguments:

- ▷ ϕ: amplitude,
- ▷ k: modulus,
- ▷ $\alpha \doteq \arcsin k$: modular angle,
- ▷ $m \doteq k^2$: parameter or modulus,
- ▷ $a,\ n = -a^2$: parameter in two common renditions.

– First kind:

$$\mathrm{F}(\phi, k) = \mathrm{F}(\phi|m) = \mathrm{F}(\phi\backslash\alpha) \doteq \int_0^\phi \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}}.$$

– Second kind:

$$\mathrm{E}(\phi, k) = \mathrm{E}(\phi|m) = \mathrm{E}(\phi\backslash\alpha) \doteq \int_0^\phi d\theta\, \sqrt{1 - k^2 \sin^2\theta}.$$

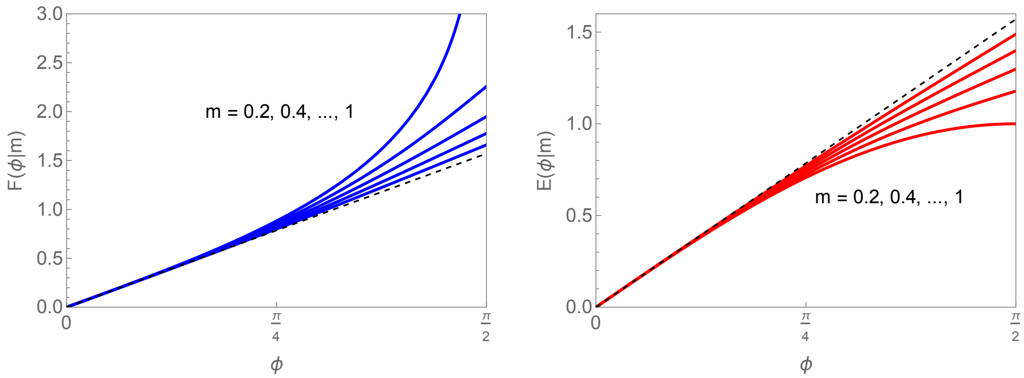
– Third kind:

$$\Pi(\phi, k, a) = \Pi(n; \phi|m) \doteq \int_0^\phi \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}\,(1 + a^2 \sin^2\theta)}.$$

Most applications in physics pertain to the modulus range $0 \leq k \leq 1$. In Mathematica these functions are named as follows:

$$\mathsf{EllipticF}[\phi, \mathsf{m}], \qquad \mathsf{EllipticE}[\phi, \mathsf{m}], \qquad \mathsf{EllipticPi}[\mathsf{n}, \phi, \mathsf{m}].$$

Graphs produced by Mathematica of the functions $\mathrm{F}(\phi|m)$ and $\mathrm{E}(\phi|m)$ versus amplitude ϕ for selected values of the modulus m are shown below.



Complete elliptic integrals:

Elliptic integrals become complete if we set $\phi = \pi/2$. The commonly used notation is, unfortunately, ambiguous. It must always be ascertained whether the argument in use is k or $m = k^2$.

– First kind:

$$\mathrm{K}(k) \doteq \mathrm{F}(\pi/2, k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}},$$

$$\mathrm{K}(m) \doteq \mathrm{F}(\pi/2|m) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - m \sin^2\theta}}.$$

– Second kind:

$$\mathrm{E}(k) \doteq \mathrm{E}(\pi/2, k) = \int_0^{\pi/2} d\theta \, \sqrt{1 - k^2 \sin^2\theta},$$

$$\mathrm{E}(m) \doteq \mathrm{E}(\pi/2|m) = \int_0^{\pi/2} d\theta \, \sqrt{1 - m \sin^2\theta}.$$

– Third kind:

$$\Pi(k, a) \doteq \Pi(\pi/2, k, a) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}\,(1 + a^2 \sin^2\theta)},$$

$$\Pi(n|m) \doteq \Pi(n; \pi/2|m) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - m \sin^2\theta}\,(1 - n \sin^2\theta)}.$$

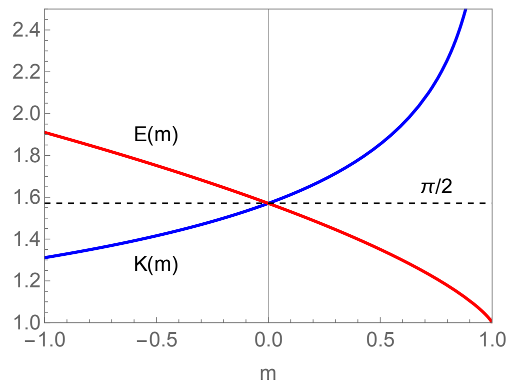
In Mathematica these functions are named as follows:

$$\mathsf{EllipticK[m]}, \qquad \mathsf{EllipticE[m]}, \qquad \mathsf{EllipticPi[n, m]}.$$

Graphs produced by Mathematica of the functions $\mathrm{K}(m)$ and $\mathrm{E}(m)$ versus m are shown below. Note the analytic continuation to negative m, which implies imaginary k.

The function $\mathrm{K}(m)$ diverges logarithmically at $m = 1$:

$$\mathrm{K}(m) \overset{m\to 1}{\rightsquigarrow} \frac{1}{2}|\ln(1-m)|.$$



Power series of complete elliptic integrals of the first and second kind are expressed in two renditions:

$$\mathrm{K}(m) = \frac{\pi}{2}\sum_{n=0}^{\infty}\begin{pmatrix} -1/2 \\ n \end{pmatrix}^2 m^n = \frac{\pi}{2}\sum_{n=0}^{\infty}\left(\frac{(2n)!}{2^{2n}(n!)^2}\right)^2 m^n$$
$$= \frac{\pi}{2}\left[1 + \frac{m}{4} + \frac{9m^2}{64} + \cdots\right],$$

$$\mathrm{E}(m) = \frac{\pi}{2}\sum_{n=0}^{\infty}\begin{pmatrix} 1/2 \\ n \end{pmatrix}\begin{pmatrix} -1/2 \\ n \end{pmatrix} m^n = \frac{\pi}{2}\sum_{n=0}^{\infty}\left(\frac{(2n)!}{2^{2n}(n!)^2}\right)^2 \frac{m^n}{1-2n}$$
$$= \frac{\pi}{2}\left[1 - \frac{m}{4} - \frac{3m^2}{64} - \cdots\right].$$

Both versions of the expansion are readily recognized by Mathematica to complete elliptic integrals.

Jacobi elliptic functions:

Inverse functions of the incomplete elliptic integral of the first kind in the following sense produce the Jacobi amplitude function,

$$\mathrm{F}(\phi|m) = u \quad \longleftrightarrow \quad \mathrm{am}(u|m) = \phi,$$

and three kinds of Jacobi elliptic functions,

$$\mathrm{F}(\phi|m) = u \quad \longleftrightarrow \quad \begin{cases} \mathrm{sn}(u|m) = \sin\phi, \\ \mathrm{cn}(u|m) = \cos\phi, \\ \mathrm{dn}(u|m) = \sqrt{1 - m\sin^2\phi}. \end{cases}$$

This notation is closest to the syntax of Mathematica:

JacobiAmplitude[u, m], JacobiSN[u, m], JacobiCN[u, m], JacobiDN[u, m].

The transcription to the notation with modulus k is straightforward:

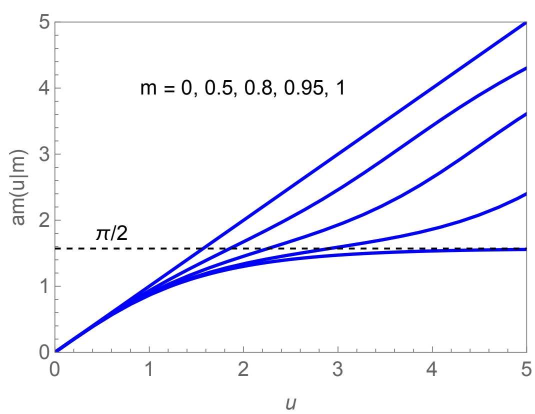
$$\mathrm{F}(\phi, k) = u \quad \longleftrightarrow \quad \begin{cases} \mathrm{am}(u, k) = \phi, \\ \mathrm{sn}(u, k) = \sin\phi, \\ \mathrm{cn}(u, k) = \cos\phi, \\ \mathrm{dn}(u, k) = \sqrt{1 - k^2\sin^2\phi}. \end{cases}$$

Jacobi amplitude function:

The amplitude function including the limiting cases,

$$\lim_{m\to 0} \mathrm{am}(u|m) = u, \quad \lim_{m\to 1} \mathrm{am}(u|m) = \arctan\left(\tanh\frac{u}{2}\right).$$

is plotted below for various values of m:



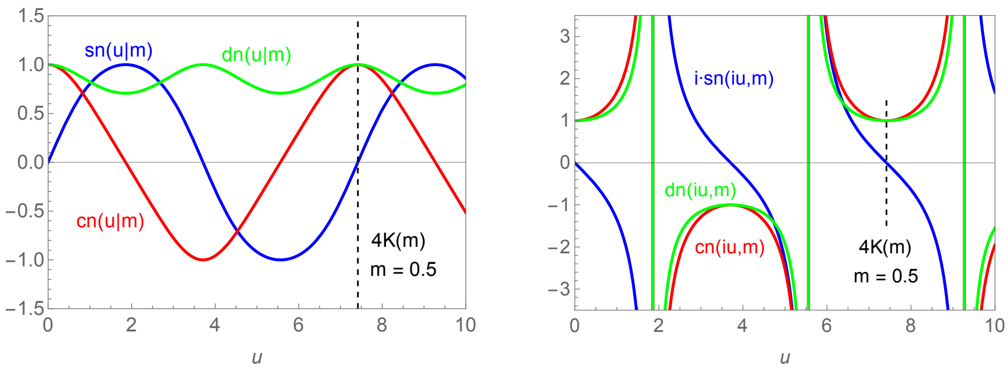
Jacobi sn, cn, and dn function:

Jacobi elliptic functions are doubly periodic generalizations of circular (trigonometric) and hyperbolic functions.

Circular functions are periodic for real variables and hyperbolic functions for imaginary variables with period 2π [gmd7-A]:

$$\sin(\imath u) = \imath \sinh u, \quad \cos(\imath u) = \cosh u,$$
$$\sinh(\imath u) = \imath \sin u, \quad \cosh(\imath u) = \cos u.$$

Jacobi elliptic functions are periodic for real and imaginary variables with period $4\mathrm{K}(m)$ for sn and cn and period $2\mathrm{K}(m)$ for dn.



– Circular limit for real variables:

$$\lim_{m\to 0} \mathrm{sn}(u|m) = \sin u, \quad \lim_{m\to 0} \mathrm{cn}(u|m) = \cos u, \quad \lim_{m\to 0} \mathrm{dn}(u|m) = 1.$$

– Hyperbolic limit for real variables:

$$\lim_{m\to 1} \mathrm{sn}(u|m) = \tanh u, \quad \lim_{m\to 1} \mathrm{cn}(u|m) = \mathrm{sech}\, u, \quad \lim_{m\to 1} \mathrm{dn}(u|m) = \mathrm{sech}\, u.$$

– Circular limit for imaginary variables:

$$\lim_{m\to 1} \mathrm{sn}(\imath u|m) = \imath \tan u, \quad \lim_{m\to 1} \mathrm{cn}(\imath u|m) = \sec u, \quad \lim_{m\to 1} \mathrm{dn}(\imath u|m) = \sec u.$$

– Hyperbolic limit for imaginary variables:

$$\lim_{m\to 0} \mathrm{sn}(\imath u|m) = \imath \sinh u, \quad \lim_{m\to 0} \mathrm{cn}(\imath u|m) = \cosh u, \quad \lim_{m\to 0} \mathrm{dn}(\imath u|m) = 1.$$

Generalization of $\sin^2 u + \cos^2 u = 1$ and $\cosh^2 u - \sinh^2 u = 1$:

$$\begin{aligned}
&\mathrm{sn}^2(u|m) + \mathrm{cn}^2(u|m) = 1, \\
&m\,\mathrm{sn}^2(u|m) + \mathrm{dn}^2(u|m) = 1, \\
&\mathrm{dn}^2(u|m) - m\,\mathrm{cn}^2(u|m) = 1 - m.
\end{aligned}$$

Generalization of derivatives for circular and hyperbolic functions:

$$\begin{aligned}
&\frac{d}{du}\,\mathrm{am}(u|m) = \mathrm{dn}(u|m), \\
&\frac{d}{du}\,\mathrm{sn}(u|m) = \mathrm{cn}(u|m)\mathrm{dn}(u|m), \\
&\frac{d}{du}\,\mathrm{cn}(u|m) = -\mathrm{sn}(u|m)\mathrm{dn}(u|m), \\
&\frac{d}{du}\,\mathrm{dn}(u|m) = -m\,\mathrm{sn}(u|m)\mathrm{cn}(u|m).
\end{aligned}$$

Relations which accommodate a modulus out of standard range:

$$\begin{aligned}
&\mathrm{cn}(u|k^2) = \mathrm{dn}(ku, k^{-2}), \\
&\mathrm{sn}(u|k^2) = k^{-1}\mathrm{sn}(ku, k^{-2}), \\
&\mathrm{dn}(u|k^2) = \mathrm{cn}(ku, k^{-2}).
\end{aligned}$$

Generalization of trigonometric angle-sum relations:

$$\begin{aligned}
\mathrm{sn}(u+v|m) &= \frac{\mathrm{sn}(u|m)\mathrm{cn}(v|m)\mathrm{dn}(v|m) + \mathrm{sn}(v|m)\mathrm{cn}(u|m)\mathrm{dn}(u|m)}{1 - m\,\mathrm{sn}^2(u|m)\mathrm{sn}^2(v|m)}, \\
\mathrm{cn}(u+v|m) &= \frac{\mathrm{cn}(u|m)\mathrm{cn}(v|m) - \mathrm{sn}(u|m)\mathrm{dn}(u|m)\mathrm{sn}(v|m)\mathrm{dn}(v|m)}{1 - m\,\mathrm{sn}^2(u|m)\mathrm{sn}^2(v|m)}, \\
\mathrm{dn}(u+v|m) &= \frac{\mathrm{dn}(u|m)\mathrm{dn}(v|m) - m\,\mathrm{sn}(u|m)\mathrm{cn}(u|m)\mathrm{sn}(v|m)\mathrm{cn}(u|m)}{1 - m\,\mathrm{sn}^2(u|m)\mathrm{sn}^2(v|m)}.
\end{aligned}$$

Plane pendulum:

Elliptic integrals and elliptic functions feature prominently in the exact analysis of the plane pendulum.

A point mass m is constrained by a massless rod to move in a vertical circle of radius l in a uniform gravitational field g. The dynamical variable of choice is the angle $\theta(t)$ of the rod away from the downward orientation.

The Lagrange equation of this system,

$$\ddot{\theta} + \omega_0^2 \sin\theta = 0, \quad \omega_0 = \sqrt{\frac{g}{l}},$$

is a nonlinear 2$^{\text{nd}}$-order ODE whose general solution can be expressed as the Jacobi amplitude function with two integration constants [gex145]:

$$\theta(t) = \pm 2\text{am}\left(\frac{1}{2}\sqrt{(2\omega_0^2 + c_1)(t + c_2)^2} \,\Big|\, \frac{4\omega_0^2}{2\omega_0^2 + c_1}\right).$$

Traditional analytic solutions start with the 1$^{\text{st}}$-order ODE inferred from the statement of energy conservation $E(\theta, \dot{\theta}) = \text{const}$ (first integral). One-parameter solutions for oscillatory motion [gex10],

$$\theta(t) = 2\arcsin\Big(k\,\text{sn}(\omega_0 t, k)\Big), \quad 0 < k \doteq \sqrt{\frac{E}{2mgl}} < 1,$$

and rotational motion [gex11],

$$\theta(t) = 2\arcsin\Big(\text{sn}(\omega_0 t/\kappa, \kappa)\Big), \quad 0 < \kappa \doteq \sqrt{\frac{2mgl}{E}} < 1,$$

are thus established. The period of oscillation and rotation are

$$\tau_{\text{osc}} = \frac{4}{\omega_0}\,\text{K}(k), \quad \tau_{\text{rot}} = \frac{2\kappa}{\omega_0}\,\text{K}(k).$$

In addition to these two types of periodic solutions there is the aperiodic separatrix motion for $E = 2mgl$:

$$\theta(t) \to 2\arcsin\Big(\tanh(\omega_0 t)\Big) \overset{t\to\infty}{\rightsquigarrow} \pi.$$

This exact analysis along these lines can be extended to action-angle coordinates, which opens the door to the analysis of the quantum pendulum.

Asymmetric top:

The free tumbling motion of an asymmetric top such as realized by a satellite in orbit is governed by Euler's equations (a topic of PHY520):

$$I_1\dot{\omega}_1 = \omega_2\omega_3(I_2 - I_3), \quad I_2\dot{\omega}_2 = \omega_3\omega_1(I_3 - I_1), \quad I_3\dot{\omega}_3 = \omega_1\omega_2(I_1 - I_2).$$

The three coupled nonlinear 1^{st}-order ODEs govern the rotational motion of this rigid body relative to a body coordinate system with origin at the center of mass and axes in the principal directions.

The inertia tensor of any rigid body is symmetric. It can be diagonalized by an orthogonal transformation (to principal axes). The elements I_i of the diagonalized inertia tensor are the principal moments of inertia.

The solution of Euler's equations yields the (instantaneous) angular velocities $\omega_i(t)$ of rotation about the principal axes. The vector $\boldsymbol{\omega}(t)$ changes direction and magnitude in the body coordinate system.

The analytic solution assumes $I_1 < I_2 < I_3$ without loss of generality.

- Inverse moments of inertia: $J_i \doteq 1/I_i$.
- Angular momentum components:[1] $L_i = I_i\omega_i$.
- Euler's equations transformed:[2] $\dot{L}_i = L_j L_k(J_k - J_j)$.
- Periodic solution expressed as Jacobi elliptic functions:

$$L_1(t) = a_1\text{dn}(\Omega t, k), \quad L_2(t) = a_2\text{sn}(\Omega t, k), \quad L_3(t) = a_3\text{cn}(\Omega t, k).$$

- Integration constants: Energy E, angular momentum L (magnitude), and phase angle (here set to zero).
- Range of energy for physical solutions: $J_3L^2 < 2E < J_1L^2$.
- Amplitudes: $a_1^2 = \dfrac{2E - J_3L^2}{J_1 - J_3}, \quad a_2^2 = \dfrac{J_1L^2 - 2E}{J_1 - J_2}, \quad a_3^2 = \dfrac{J_1L^2 - 2E}{J_1 - J_3}.$
- Modulus: $k^2 = \dfrac{J_2 - J_3}{J_1 - J_2}\,\dfrac{J_1L^2 - 2E}{2E - J_3L^2}.$
- Frequency scale: $\Omega^2 = (J_1 - J_2)(2E - J_3L^2)$.
- Period of the motion: $T = 4\text{K}(k)/\Omega$.

The transformation of this solution from the body-coordinate system to an inertial coordinate system involves the solution of another set of three coupled 1^{st}-order ODEs.

[1]Note that the vectors $\mathbf{L}$ and $\boldsymbol{\omega}$ are not, in general, parallel.

[2]The indices i, j, k run over cyclic permutations $\{1,2,3\}, \{2,3,1\}, \{3,1,2\}$.