Error Function [gmd4B]

Error functions commonly appear in probability theory and, unsurprisingly, in error analysis, but also, unexpectedly, in other areas. Three variants are

– the (standard) error function:

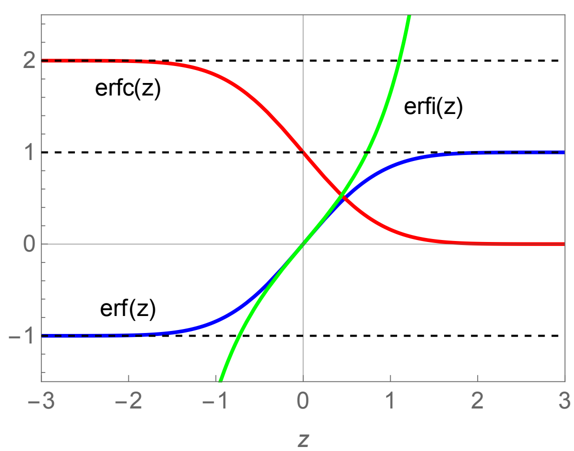
$$\mathrm{erf}(z) \doteq \frac{2}{\sqrt{\pi}} \int_0^z dt\, e^{-t^2}, \tag{1}$$

– the complementary error function:

$$\mathrm{erfc}(z) \doteq 1 - \mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty dt\, e^{-t^2}, \tag{2}$$

– the imaginary error function:

$$\mathrm{erf}(z) \doteq \frac{1}{\imath}\, \mathrm{erf}(\imath z) = \frac{2}{\sqrt{\pi}} \int_0^z dt\, e^{t^2}. \tag{3}$$



All three are entire functions and real for z along the real axis.

The two antisymmetric error functions have identical power series except for sign alternations in one of them:

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^z dt\, t^{2n} = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{z^{2n+1}}{2n+1}. \tag{4}$$

Error functions have a wide range of applications, notably in probability theory [gmd9] and in diffusion processes, a PDF application [gmd11].

Fresnel integrals:

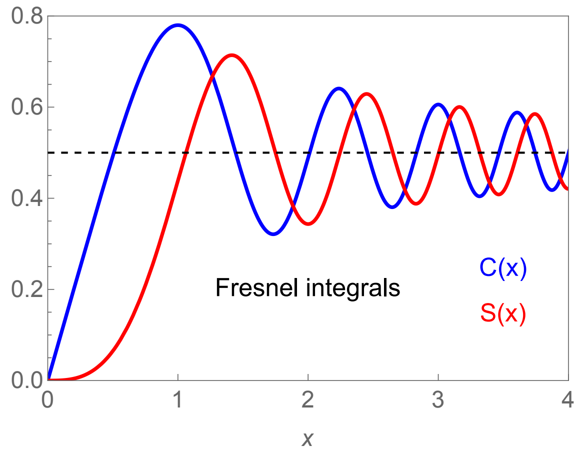
Fresnel integrals, defined as

$$\mathrm{C}(z) \doteq \int_0^z dt \cos\left(\frac{\pi}{2}\, t^2\right), \quad \mathrm{S}(z) \doteq \int_0^z dt \sin\left(\frac{\pi}{2}\, t^2\right), \tag{5}$$

have applications in the theories of diffraction and vibrations.

Symmetry: $\mathrm{C}(-z) = -\mathrm{C}(z), \quad \mathrm{S}(-z) = -\mathrm{S}(z)$.

Limits: $\lim_{x\to\infty} \mathrm{C}(x) = \frac{1}{2} = \lim_{x\to\infty} \mathrm{S}(x), \quad \lim_{x\to\infty} \mathrm{C}(\imath x) = \frac{\imath}{2} = -\lim_{x\to\infty} \mathrm{S}(ix)$.



Power series:

$$\mathrm{C}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (\pi/2)^{2n}}{(2n)!(4n+1)}\, z^{4n+1} = z - \frac{\pi^2}{40}\, z^5 + \frac{\pi^4}{3456}\, z^9 - \cdots \tag{6}$$

$$\mathrm{S}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (\pi/2)^{2n+1}}{(2n+1)!(4n+3)}\, z^{4n+3} = \frac{\pi}{6}\, z^3 - \frac{\pi^3}{336}\, z^7 + \frac{\pi^5}{42240}\, z^{11} - \cdots \tag{7}$$

Note the high order of the first correction. The linear rise of $\mathrm{C}(z)$ and the cubic rise of $\mathrm{S}(z)$ remain accurate across a significant interval.

Fresnel integrals are related to error functions via [gex95]:

$$\mathrm{erf}\left(\sqrt{\imath}\, z\right) = (1+\imath)\left[\mathrm{C}\left(\sqrt{\frac{2}{\pi}}\, z\right) - \imath \mathrm{S}\left(\sqrt{\frac{2}{\pi}}\, z\right)\right], \tag{8}$$

$$\mathrm{erf}\left(\sqrt{-\imath}\, z\right) = (1-\imath)\left[\mathrm{C}\left(\sqrt{\frac{2}{\pi}}\, z\right) + \imath \mathrm{S}\left(\sqrt{\frac{2}{\pi}}\, z\right)\right] \tag{9}$$

One commonality of the functions discussed in [gmd4A] and here in [gmd4B] is that heir primary definition involves integrals of more elementary functions or products thereof.

Other special functions of no less importance (to be discussed in separate modules) are defined primarily as solutions of differential equations.

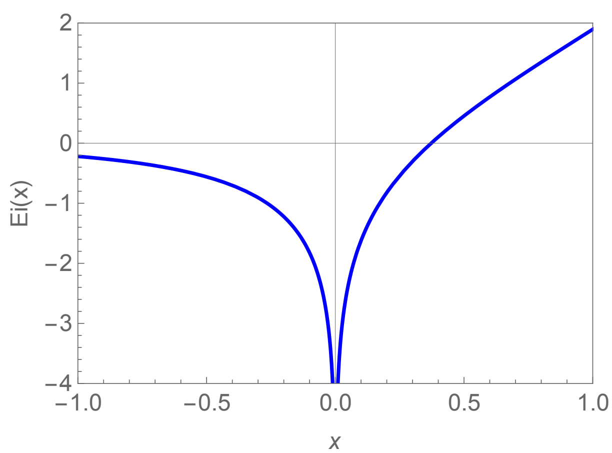
Here we continue with brief portraits of a additional special functions which have the word *integral* in their name.

Exponential integral:

The defining integral of this function, here expressed in two versions,

$$\mathrm{Ei}(z) \doteq \int_{-\infty}^{z} dt\, \frac{e^t}{t} = -\int_{-z}^{\infty} dt \frac{e^{-t}}{t}, \tag{10}$$

involves, for positive real z, an integration across a nonintegrable divergence. This integration is to be interpreted as principal-value integral (a term familiar from complex analysis [gmd7]).



Salient properties of the exponential integral for real z:

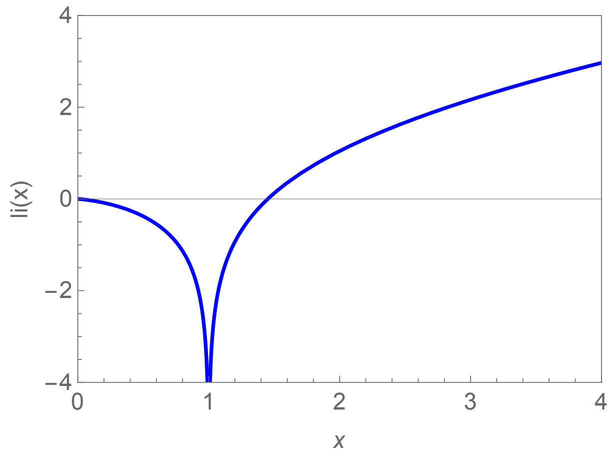
- Divergent singularity: $\mathrm{Ei}(x) \stackrel{x\to 0}{\longrightarrow} \ln|x|$.
- Series representation: $\mathrm{Ei}(x) = \gamma + \ln|x| + \sum_{n=1}^{\infty} \frac{x^n}{n!n}$.
- Location of zero: $\mathrm{Ei}(x_0) = 0$ at $x_0 = 0.372507....$
- Asymptotics: $\mathrm{Ei}(x) \stackrel{x\to\pm\infty}{\longrightarrow} \frac{e^x}{x}$.
- Relation to Gamma function: $\mathrm{Ei}(x) = -\Gamma(0, x)$.

Logarithmic integral:

The logarithmic integral is closely related to the exponential integral:

$$\mathrm{li}(z) \doteq \int_0^z \frac{dt}{\ln t} = \int_{-\infty}^{\ln z} du \, \frac{e^u}{u}. \tag{11}$$

The relation is evident in the second integral, which uses $u = \ln t$.



- The logarithmic integral is a real function at $z \geq 0$ for real z.
- The explicit relation to the exponetial relation reads $\mathrm{li}(x) = \mathrm{Ei}(\ln x)$.
- The zeros of $\mathrm{li}(x)$ at $x = 0$ and $x = 1.45137\ldots$ map onto the zeros of $\mathrm{Ei}(x')$ at $x' = -\infty$ and $x' = 0.372507\ldots$ via $x = e^{x'}$.
- The weak divergence of $\mathrm{Ei}(x') \sim \ln|x'|$ at $x' = 0$ maps onto the yet weaker singularity $\mathrm{li}(x) \sim \ln|\ln x|$ at $x = 1$.
- The zero of $\mathrm{li}(x)$ at $x = 0$ is an essential singularity. Expansions at such points are hazardous.
- A convergent series representation of $\mathrm{li}(x)$ can be inferred from that found for $\mathrm{Ei}(x)$ via mapping:

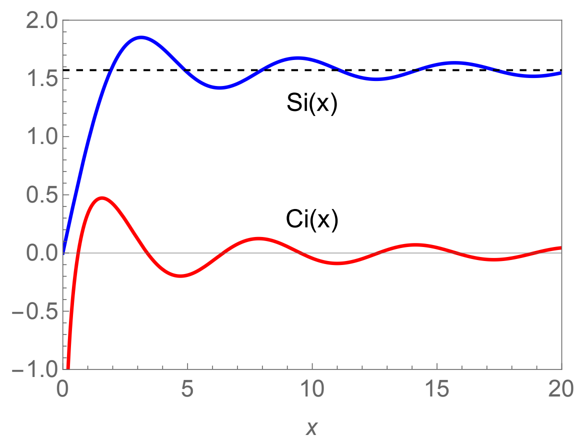
$$\mathrm{li}(x) = \gamma + \ln|\ln x| + \sum_{n=1}^{\infty} \frac{(\ln x)^n}{n!n} \quad : \; x > 0.$$

Sine and cosine integrals:

These functions, defined by the integrals (here for points on the real axis),

$$\mathrm{Si}(x) = \int_0^x dt\, \frac{\sin t}{t}, \quad \mathrm{Ci}(x) = \int_\infty^x dt\, \frac{\cos t}{t} \quad : \ x > 0, \tag{12}$$

show up occasionally in physics applications too.



– Behavior near $x = 0$:

$$\mathrm{Si}(x) = x - \frac{x^3}{18} + \frac{x^5}{600} + \mathrm{O}\left(x^6\right),$$
$$\mathrm{Ci}(x) = \ln x + \gamma - \frac{x^2}{4} + \frac{x^4}{96} + \mathrm{O}\left(x^6\right).$$

– Weakly damped oscillations for $x \to \infty$:

$$\mathrm{Si}(x) \sim \frac{\pi}{2} - \frac{\cos x}{x},$$
$$\mathrm{Si}(x) \sim -\frac{\cos x}{x^2}.$$