Gamma Function [gmd4A]

The interpolation of factorials,

$$n! \doteq \prod_{l=1}^{n} l = 1 \cdot 2 \cdots (n-1)n \quad : \; n \in \mathbb{N} \tag{1}$$

to include real numbers, and the extension to complex numbers $z \in \mathbb{C}$ lead to the *Gamma function*,

$$\Gamma(z) \doteq \int_0^\infty dt\, e^{-t} t^{z-1}, \tag{2}$$

which is analytic for $\Re[z] > 0$. The recurrence relation,

$$\Gamma(z+1) = z\Gamma(z), \tag{3}$$

follows directly from an integration by parts [gex2]. The repeated use of the recurrence relation (3) with $\Gamma(1) = 1$ establishes the relation,

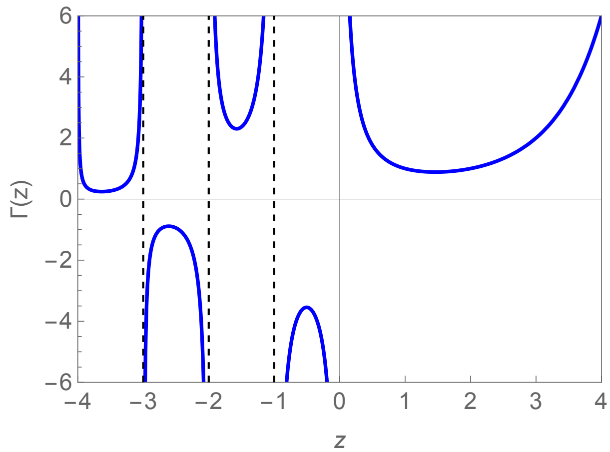
$$\Gamma(n+1) = n! \quad : \; n \in \mathbb{N}. \tag{4}$$

The analytic continuation of (2) to $\Re[z] \leq 0$ can be accomplished by splitting the integral into two parts as follows:

$$\begin{aligned} \Gamma(z) &= \int_0^1 dt\, e^{-t} t^{z-1} + \int_1^\infty dt\, e^{-t} t^{z-1} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 dt\, t^{z+n-1} + \int_1^\infty dt\, e^{-t} t^{z-1} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{z+n} + \int_1^\infty dt\, e^{-t} t^{z-1}. \end{aligned} \tag{5}$$

The last integral converges everywhere. The divergences at non-positive integer values of z are first-order poles.

The integral representation (2) opens the door to any number of variable substitutions which then relate many definite integrals to Gamma functions.



The *Weierstrass definition* of the Gamma function in the form of an infinite product is consistent with (2), but covers a more general range of the complex variable:

$$\Gamma(z) \doteq \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \frac{e^{z/n}}{1+z/n} \quad : \; z \notin \{0, -1, -2, \ldots\}, \tag{6}$$

where the *Euler constant* is

$$\gamma \doteq \lim_{N\to\infty} \left(\sum_{n=1}^{N} \frac{1}{n} - \ln N \right) = -\Gamma'(1) = 0.57721\ldots \tag{7}$$

The recurrence relation (3) can also be demonstrated for (6) [gex2].

Gamma functions at half-integer values are a common occurrence:

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!}{2^{2n} n!}\sqrt{\pi} = \frac{(2n-1)!!}{2^n}\sqrt{\pi}. \tag{8}$$

Double factorials for even and odd integers are defined as $(2n)!! = 2\cdot 4\cdot 6 \cdots 2n$ and $(2n-1)!! = 1\cdot 3\cdot 5 \cdots (2n-1)$. These definitions can also be interpolated and extended to complex variables.

Stirling's asymptotic formula is commonly used in processing combinatorial results for use statistical mechanics of macroscopic systems (large n):

$$\Gamma(n+1) \rightsquigarrow \sqrt{2\pi n}\, n^n e^{-n}. \tag{9}$$

Beta function:

A direct descendent of the Gamma function is the *Beta function*:

$$\mathrm{B}(m,n) \doteq \int_0^1 dx\, x^{m-1}(1-x)^{n-1} = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \quad : \; m>0,\; n>0. \tag{10}$$

It is related to definite integrals of powers of trigonometric functions commonly occurring in physics. The substitution, $x = \sin^2\theta$, for example, directly yields

$$\mathrm{B}(m,n) = 2\int_0^{\pi/2} d\theta [\sin\theta]^{2m-1}[\cos\theta]^{2n-1}. \tag{11}$$

One way of establishing the relation of the Beta function to the Gamma function takes advantage of this trigonometric representation [gex3]. The result is

$$\mathrm{B}(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}. \tag{12}$$

Two important relations between Gamma functions,

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}, \tag{13}$$

$$2^{2x-1}\Gamma(x)\Gamma\left(x+\tfrac{1}{2}\right) = \sqrt{\pi}\,\Gamma(2x), \tag{14}$$

are efficiently proven by way of the Beta function. The second relation is known as *duplication formula* [gex39].

Binomial series:

Binomial series:

$$(1+x)^a = \sum_{k=0}^{\infty} \binom{a}{k} x^k \;\stackrel{a\to n}{\longrightarrow}\; \sum_{k=0}^{n} \binom{n}{k} x^k. \tag{15}$$

Binomial coefficients:

$$\binom{a}{k} = \frac{\Gamma(a+1)}{\Gamma(k+1)\Gamma(a-k+1)} \;\stackrel{a\to n}{\longrightarrow}\; \frac{n!}{k!(n-k)!}. \tag{16}$$

The divergent $\Gamma(n-k+1)$ for $k>n$ terminates the series to a finite sum.

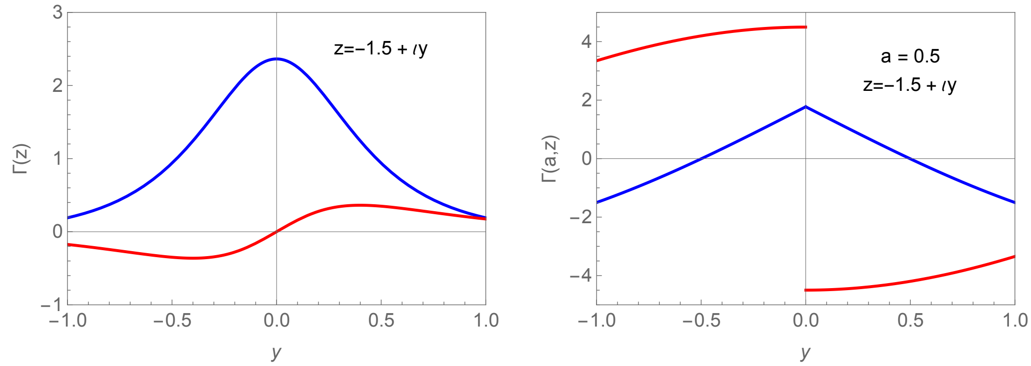
Incomplete Gamma functions:

The incomplete Gamma functions,

$$\Gamma(a,z) \doteq \int_z^\infty dt e^{-t} t^{a-1}, \quad \gamma(a,z) \doteq \int_0^z dt e^{-t} t^{a-1}, \tag{17}$$

are complementary to each other: $\Gamma(a,z) + \gamma(a,z) = \Gamma(z)$. $\Gamma(a,z)$ has a branch cut discontinuity along the negative real axis in the z-plane.

The graphs below show the real part (blue) and the imaginary part (red) of the complete (left) and incomplete (right) Gamma functions for an interval of z that cuts across the negative real axis.



$\Gamma(z)$ is smooth along any path which avoids non-positve integer values on the real axis. By contrast, $\Gamma(a,z)$ has a discontinuous imaginary part and a real part with a discontinuity in slope.

The relation between incomplete and complete Gamma functions is subtle. It is the parameter a of the former which becomes the variable of the latter:

$$\lim_{z\to 0} \Gamma(a,z) = \lim_{z\to\infty} \gamma(a,z) = \Gamma(a). \tag{18}$$

The power series,

$$\gamma(a,z) = z^a \sum_{n=0}^\infty \frac{(-1)^n z^n}{n!(n+a)} \quad : \ a = 0, \tag{19}$$

can be obtained by expanding e^{-t} under the defining integral.

With repeated integrations by part applied to the defining expression, we obtain the (non-convergent) asymptotic expansion,

$$\Gamma(a,z) \overset{z\to\infty}{\rightsquigarrow} \Gamma(a) z^{a-1} e^{-z} \sum_{k=0}^\infty \frac{1}{\Gamma(a-k) z^k}, \tag{20}$$

which turns into a finite sum for $a = n+1 = 1,2,3,\ldots$

Polygamma functions:

The basic Polygamma function is the Digamma function, also known under the name psi function (here considered for real arguments):

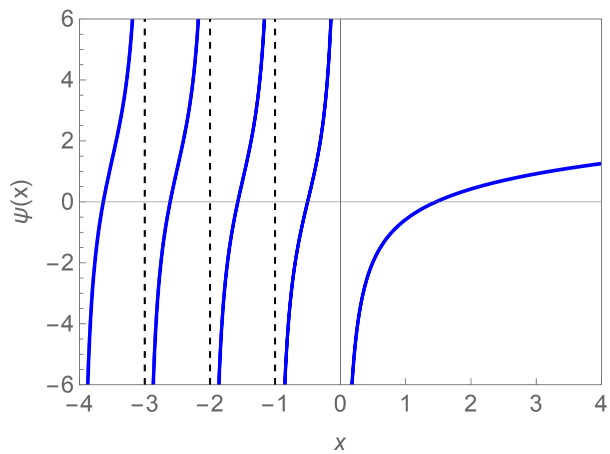
$$\psi(x) \doteq \frac{d}{dx} \ln \Gamma(x) \quad : \; x \neq 0, -1, -2, \ldots \tag{21}$$

The series representation,

$$\psi(x) = -\gamma + \sum_{n=0}^{\infty} \left[\frac{1}{n+1} - \frac{1}{x+n} \right] \quad : \; x > 0, \tag{22}$$

can be extracted from the Weierstrass product representation (6) of $\Gamma(x)$. The recurrence relation (3) for $\Gamma(x)$ transcribes as follows [gex92]:

$$\psi(x+1) = \psi(x) + \frac{1}{x}. \tag{23}$$



The integral representation of $\psi(x)$ derived from that of $\Gamma(x)$ reads

$$\psi(x+1) = \int_0^{\infty} dt \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{e^t - 1} \right), \tag{24}$$

where the separate integral of each term would diverge. The asymptotic series (with Bernoulli numbers B_{2n}) reads

$$\psi(x+1) \stackrel{x\to\infty}{\longrightarrow} \ln x + \frac{1}{2x} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{B_{2n}}{n x^{2n}}. \tag{25}$$

This expansion is the source of Stirling's asymptotic formula (9).

Higher-order Polygamma functions are inferred from the Digamma function by differentiation:

$$\psi^{(m)}(x) = \frac{d^{m+1}}{dx^{m+1}} \ln \Gamma(x) \quad : \; m = 1, 2, \ldots \tag{26}$$

The series representation derived from (25) reads [gex93]

$$\psi^{(m)}(x) = (-1)^{m+1} m! \sum_{n=0}^{\infty} \frac{1}{(n+x)^{m+1}} \quad : \; x > 0, \quad m = 1, 2, \ldots . \tag{27}$$

Zeta function:

The Riemann Zeta function,

$$\zeta(q) \doteq \sum_{n=1}^{\infty} \frac{1}{n^q} \quad : \; q > 1, \tag{28}$$

is related to Polygamma functions, which is most evident in the following relation to their series representation (27):

$$\begin{aligned} \psi^{(m)}(1) &= (-1)^{m+1} m! \sum_{n=0}^{\infty} \frac{1}{(n+1)^{m+1}} = (-1)^{m+1} m! \sum_{n=1}^{\infty} \frac{1}{n^{m+1}} \\ &= (-1)^{m+1} m! \, \zeta(m+1) \quad : \; m = 1, 2, 3, \ldots \end{aligned} \tag{29}$$

The function $\zeta(q)$ has many applications in physics. It is also important in number theory, as is manifest, for example, in the following representation featuring prime numbers:

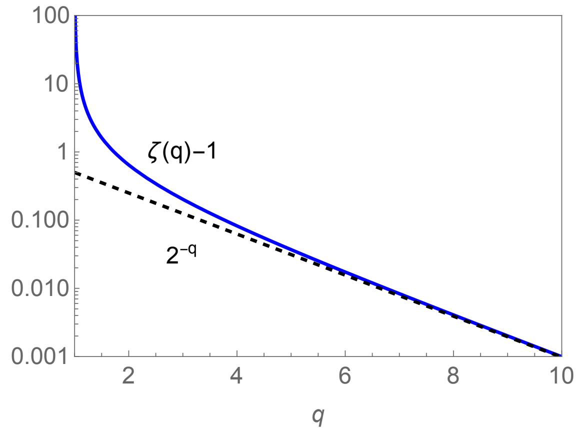
$$\zeta(q) = \prod_P \left(1 - P^{-q}\right)^{-1} \quad : \; P \in \text{ prime numbers.} \tag{30}$$

$$\begin{aligned} \zeta(q)\left(1 - 2^{-q}\right) &= 1 + 2^{-q} + 3^{-q} + 4^{-q} + \cdots - \left(2^{-q} + 4^{-q} + 6^{-q} + \cdots\right) \\ &= 1 + 3^{-q} + 5^{-q} + 7^{-q} + \cdots \end{aligned}$$

$$\begin{aligned} \zeta(q)\left(1 - 2^{-q}\right)\left(1 - 3^{-q}\right) =& 1 + 3^{-q} + 5^{-q} + 7^{-q} + \cdots \\ & - \left(3^{-q} + 9^{-q} + 27^{-q} + \cdots\right) \\ =& \; 1 + 5^{-q} + 7^{-q} + 11^{-q} + \cdots \end{aligned}$$

Asymptotic behavior:

$$\zeta(q) \quad \stackrel{q\to\infty}{\longrightarrow} \quad 1+2^{-q}. \tag{31}$$



Integral representation:

$$\zeta(q) = \frac{1}{\Gamma(q)} \int_0^\infty dt\, \frac{t^{q-1}}{e^t - 1} \quad : \; q > 1. \tag{32}$$

$\triangleright$ $\displaystyle\int_0^\infty dt\, t^{q-1} e^{-nt} \stackrel{nt=x}{=} n^{-q} \int_0^\infty dx\, x^{q-1} e^{-x} = n^{-q}\Gamma(q).$

$\triangleright$ $\displaystyle\frac{1}{n^q} = \frac{1}{\Gamma(q)} \int_0^\infty dt\, t^{q-1} e^{-nt}.$

$\triangleright$ $\displaystyle\zeta(q) = \sum_{n=1}^\infty \frac{1}{n^q} = \frac{1}{\Gamma(q)} \int_0^\infty dt\, t^{q-1} \sum_{n=1}^\infty e^{-nt}.$

$\triangleright$ $\displaystyle\sum_{n=1}^\infty e^{-nt} = \sum_{n=1}^\infty \left(e^{-t}\right)^n = \frac{1}{1-e^{-t}} - 1 = \frac{1}{e^t - 1}.$

Riemann formula:

$$\frac{\zeta(1-q)}{\zeta(q)} = 2^{1-q}\pi^{-q}\Gamma(q)\cos\left(\frac{\pi q}{2}\right). \tag{33}$$

A straightforward generalization of the integral representation of $\zeta(n)$ leads to the polylogarithmic functions $\mathrm{Li}_n(z)$.

These functions play key roles in the quantum statistics of nonrelativistic bosons and fermions.

Polylogarithmic functions:

Bose-Einstein functions are polylogarithm functions:

$$g_n(z) = \mathrm{Li}_n(z) \doteq \frac{1}{\Gamma(n)} \int_0^\infty \frac{dx\, x^{n-1}}{z^{-1}e^x - 1} = \sum_{l=1}^\infty \frac{z^l}{l^n}, \qquad 0 \leq z \leq 1. \tag{34}$$

Special cases:

$$g_0(z) = \frac{z}{1-z}, \quad g_1(z) = -\ln(1-z), \quad g_\infty(z) = z. \tag{35}$$

Riemann zeta function recovered:

$$g_n(1) = \zeta(n) \doteq \sum_{l=1}^\infty \frac{1}{l^n}. \tag{36}$$

Special values:

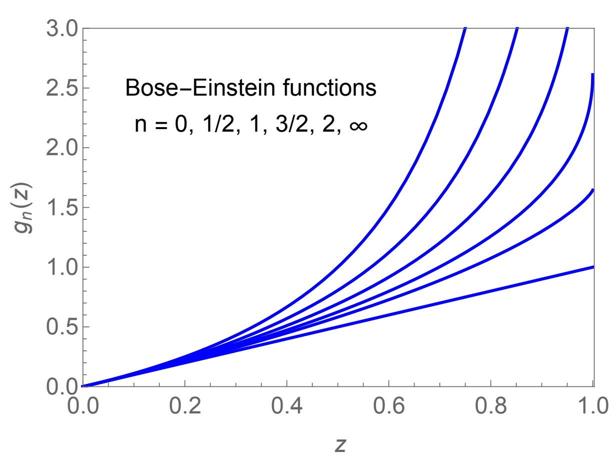
$$\zeta(1) \to \infty, \quad \zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}. \tag{37}$$

Recurrence relation:

$$z g_n'(z) = g_{n-1}(z), \qquad n \geq 1. \tag{38}$$

Singularity at $z = 1$ for non-integer n:

$$g_n(\alpha) = \Gamma(1-n)\alpha^{n-1} + \sum_{\ell=0}^\infty \frac{(-1)^\ell}{\ell!}\zeta(n-\ell)\alpha^\ell, \qquad \alpha \doteq -\ln z. \tag{39}$$



Fermi-Dirac functions are the same polylogarithm functions in a different range of the variable z (now renamed $-z$):

$$f_n(z) = -\mathrm{Li}_n(-z) \doteq \frac{1}{\Gamma(n)} \int_0^\infty \frac{dx\, x^{n-1}}{z^{-1}e^x + 1}, \quad 0 \leq z < \infty. \tag{40}$$

Series expansion:

$$f_n(z) = \sum_{l=1}^\infty (-1)^{l-1} \frac{z^l}{l^n}, \quad 0 \leq z \leq 1. \tag{41}$$

Special cases:

$$f_0(z) = \frac{z}{1+z}, \qquad f_1(z) = \ln(1+z), \qquad f_\infty(z) = z. \tag{42}$$

Recurrence relation:

$$z f_n'(z) = f_{n-1}(z), \quad n \geq 1. \tag{43}$$

Asymptotic expansion for $z \gg 1$:

$$\begin{aligned} f_n(z) &= \frac{(\ln z)^n}{\Gamma(n+1)} \left[1 + \sum_{k=2,4,\ldots} 2n(n-1)\cdots(n-k+1)\left(1 - \frac{1}{2^{k-1}}\right) \frac{\zeta(k)}{(\ln z)^k} \right] \\ &= \frac{(\ln z)^n}{\Gamma(n+1)} \left[1 + n(n-1)\frac{\pi^2}{6}(\ln z)^{-2} \right. \\ &\qquad \left. + n(n-1)(n-3)\frac{7\pi^4}{360}(\ln z)^{-4} + \ldots \right] \end{aligned} \tag{44}$$