Generalized Functions [gmd3-A]

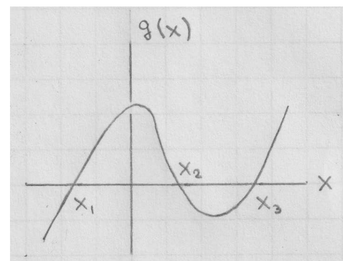
What is known in the mathematical literature as *distributions* are generalize functions with attributes not normally associated with functions. The term ‘function’ is routinely used in physics for the most common distributions.

Dirac delta function:

The generalized function known as Dirac delta $\delta(x)$ is defined by the following attributes:

$$\delta(x) = \begin{cases} 0 & : \ x \neq 0, \\ \infty & : \ x = 0, \end{cases} \qquad \int_{-\infty}^{+\infty} dx\,\delta(x) = 1.$$

Some consequences and generalizations:



- ▷ $\displaystyle\int_{-\infty}^{+\infty} dx\, f(x)\delta(x-a) = f(a).$
- ▷ $\displaystyle\int_{-\infty}^{+\infty} dx\, f(x)\delta(bx) = \frac{1}{|b|}f(0).$
- ▷ $\displaystyle\int_{-\infty}^{+\infty} dx\, f(x)\delta(bx-a) = \frac{1}{|b|}f(a/b).$
- ▷ $\displaystyle\int_{-\infty}^{+\infty} dx\, f(x)\delta\big(g(x)\big) = \sum_i \frac{1}{|g'(x_i)|}f(x_i); \quad g(x_i)=0, \quad x = x_i.$
- ▷ $\delta(\mathbf{x}-\mathbf{a}) = \delta(x-a_x)\delta(y-a_y)\delta(z-a_z).$

The delta function can be interpreted as one of k parametrized functions $\Delta_k(x, w)$ in the limit where the parameter w vanishes [lex38].

That limit does not exist for the function itself, but all attributes listed above remain valid.

The delta function is only useful as part of an integral. It is even (symmetric) and its derivative is odd (antisymmetric). The derivative of a delta function can be removed via integration by parts.

- ▷ $\displaystyle\int dx f(x)\delta(x-a) = \int dx f(x)\delta(a-x) = f(a),$
- ▷ $\displaystyle\int dx f(x)\delta'(x-a) = -\int dx f'(x)\delta(x-a) = -f'(a),$
- ▷ $\displaystyle\int dx f(x)\delta'(a-x) = \int dx f'(x)\delta(a-x) = f'(a).$