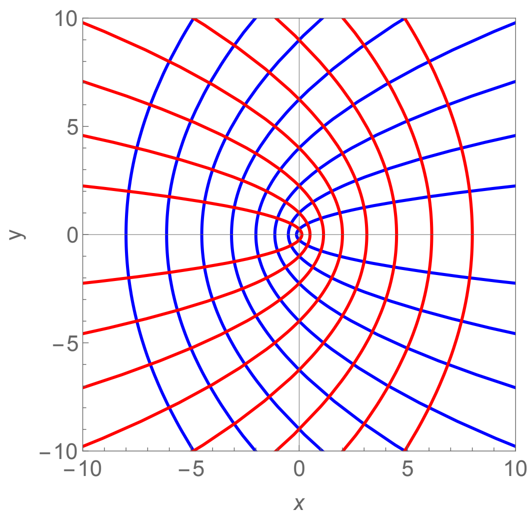
Coordinate Systems II [gmd2-B]

Parabolic cylindrical coordinates:

The part 'cylindrical' of the name indicates translational symmetry along one coordinate axis. The part 'parabolic' indicates the shape of the 2D manifolds perpendicular to the direction of translational symmetry.

– Curvilinear coordinates: u, v, z.
– Range: $-\infty < u, v, z < \infty$.
– Transformation relations to Cartesian coordinates:
$$x = \frac{1}{2}(u^2 - v^2), \quad y = uv, \quad z = z.$$
– Scale factors [gex71]: $h_u = h_v = \sqrt{u^2 + v^2}, \quad h_z = 1.$
– Transformation relations to (circular) cylindrical coordinates [gex71]:
$$u = \sqrt{2\rho}\,\cos\frac{\phi}{2}, \quad v = \sqrt{2\rho}\,\sin\frac{\phi}{2}, \quad z = z.$$
– Mutually orthogonal 2D manifolds [gex71]:
  - ▷ $z =$ const: planes perpendicular to z-axis.
  - ▷ $u =$ const: planes bent into confocal parabolic profile (red).
  - ▷ $v =$ const: planes bent into confocal parabolic profile (blue).



Blue lines: $v = 0.5, 1.0, \ldots, 4.0$ (from inside out).
Red lines: $u = 0.5, 1.0, \ldots, 4.0$ (from inside out).

Paraboloidal coordinates:

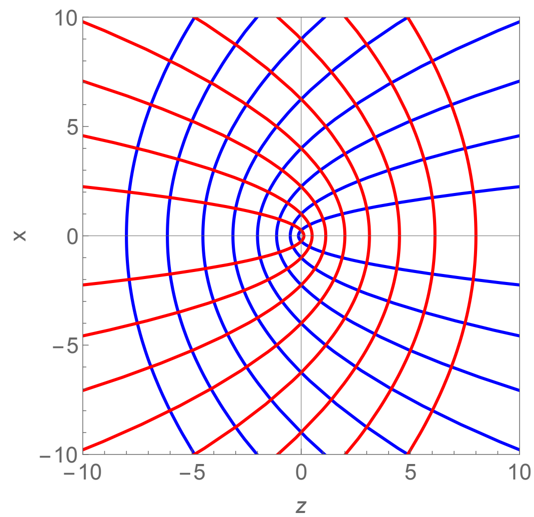
The name refers to the paraboloidal shape of two sets of 2D manifold in this set curvilinear coordinates.

The continuous translational symmetry along the z-axis present in parabolic cylindrical coordinates is replaced by a continuous rotational symmetry about the z-axis in the case of paraboloidal coordinates.

- Curvilinear coordinates: u, v, ϕ.
- Range: $u \geq 0, \quad v \geq 0, \quad 0 \leq \phi < 2\pi$.
- Transformation relations to Cartesian coordinates:

$$x = uv\cos\phi, \quad y = uv\sin\phi, \quad z = \frac{1}{2}(u^2 - v^2).$$

- Scale factors [gex72]: $h_u = h_v = \sqrt{u^2 + v^2}, \quad h_\phi = uv$.
- Mutually orthogonal 2D manifolds [gex72]:
  - ▷ $\phi =$ const: planes containing the z-axis.
  - ▷ $u =$ const: confocal paraboloidal surfaces (red).
  - ▷ $v =$ const: confocal paraboloidal surfaces (blue).



Blue lines: $v = 0.5, 1.0, \ldots, 4.0$ (from inside out).
Red lines: $u = 0.5, 1.0, \ldots, 4.0$ (from inside out).
Top half pertains to $\phi = 0$ and bottom half to $\phi = \pi$.

Elliptic cylindrical coordinates:

The continuous translational symmetry of circular cylindrical coordinates is maintained, but the continuous rotational symmetry is reduced from continuous to twofold as it was in the case of parabolic cylindrical coordinates.

Instead of two sets of mutually orthogonal surfaces with parabolic profile we now have mutually orthogonal sets of 2D manifolds with elliptic and hyperbolic profiles.

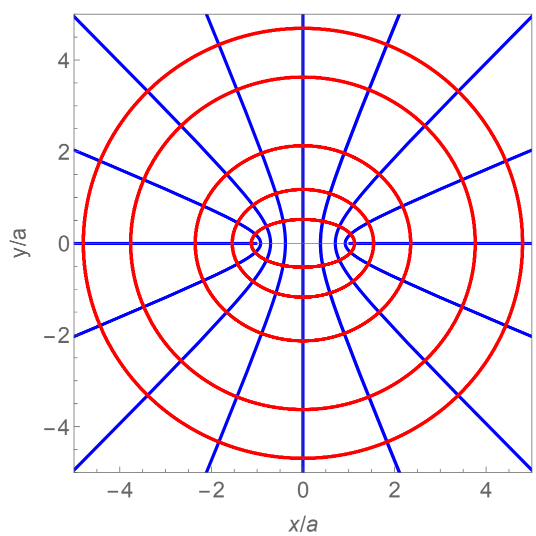
- Curvilinear coordinates: u, v, z.
- Range: $-\infty < u < \infty, \quad 0 \leq v < 2\pi, \quad -\infty < z < \infty$.
- Transformation relations to Cartesian coordinates:

$$x = a\cosh u \cos v, \quad y = a\sinh u \sin v, \quad z = z.$$

- Scale factors [gex73]: $h_u = h_v = a\sqrt{\sinh^2 u + \sin^2 v}, \quad h_z = 1$.
- Transformation relations to (circular) cylindrical coordinates [gex73]:

$$\frac{\rho^2}{a^2} = \cosh^2 u - \sin^2 v = \sinh^2 u + \cos^2 v, \quad \tan\phi = \tanh u \tan v, \quad z = z.$$

- Mutually orthogonal 2D manifolds [gex73]:
  - ▷ $z =$ const: planes perpendicular to z-axis.
  - ▷ $u =$ const: planes bent into elliptic profile (red).
  - ▷ $v =$ const: planes bent into hyperbolic profile (blue).



Blue lines: $v = n\pi/8, \quad n = 0, 1, \ldots, 8$ (from right to left).
Red lines: $u = 0.5, 1, 0, 1, 5, 2.0, 2.25$ (from inside out).

Prolate spheroidal coordinates:

Here we exchange the continuous translational symmetry of elliptic cylindrical coordinates along the z-axis by a continuous rotational symmetry about the major elliptic axis, again chosen to be the z-axis.

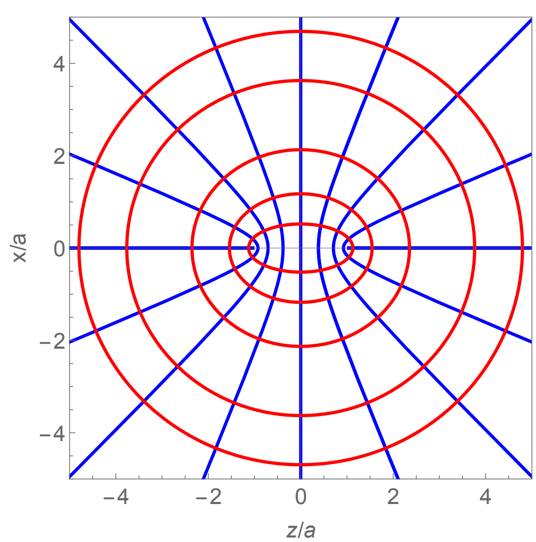
Two sets of 2D manifolds thus become mutually orthogonal ellipsoids and hyperboloids of rotation.

Spheroids are ellipsoids of rotation. Prolate means that the ellipsoidal axis of rotation is larger than the other two axes (of equal size).

- Curvilinear coordinates: ξ, η, ϕ.
- Range: $\xi \geq 0, \quad \pi/2 \leq \eta \leq \pi/2, \quad 0 \leq \phi < 2\pi$.
- Transformation relations to Cartesian coordinates:

$$x = a \sinh \xi \sin \eta \cos \phi, \quad y = a \sinh \xi \sin \eta \sin \phi, \quad z = a \cosh \xi \cos \eta.$$

- Scale factors [gex74]: $h_\xi = h_\eta = a\sqrt{\sinh^2 \xi + \sin^2 \eta}, \quad h_\phi = a \sinh \xi \sin \eta$.
- Mutually orthogonal 2D manifolds [gex74]:
  - ▷ $\phi =$ const: planes containing the z-axis.
  - ▷ $\xi =$ const: ellipsoidal surfaces of rotation (red).
  - ▷ $\eta =$ const: hyperboloidal surfaces of rotation (blue).



Blue lines: $\eta = n\pi/8, \quad n = 0, 1, \ldots, 8$ (from right to left).
Red lines: $\xi = 0.5, 1, 0, 1, 5, 2.0, 2.25$ (from inside out).
The z-axis of rotation is horizontal.

Oblate spheroidal coordinates:

Switching from prolate to oblate spheroids means making the ellipsoidal axis of rotation (the z-axis again) smaller than the other two axes (of equal size).

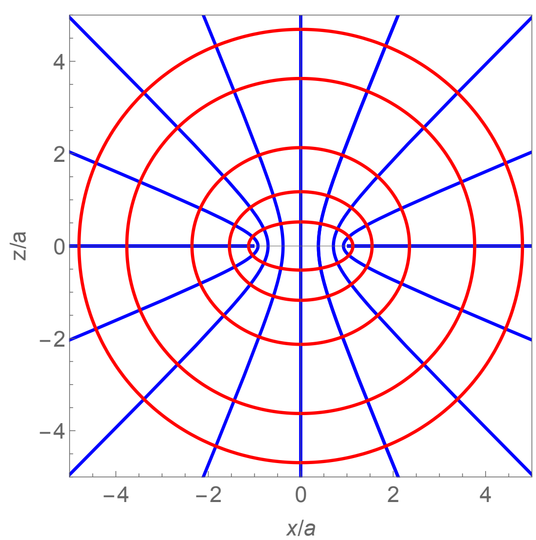
Planet Earth, which is somewhat flattened due its spinning motion, is (roughly) an oblate spheroid.

Two sets of 2D manifolds remain mutually orthogonal ellipsoids and hyperboloids of rotation as in the prolate case.

- Curvilinear coordinates: ξ, η, ϕ.
- Range: $\xi \geq 0, \quad \pi/2 \leq \eta \leq \pi/2, \quad 0 \leq \phi < 2\pi$.
- Transformation relations to Cartesian coordinates:

$$x = a\cosh\xi\cos\eta\cos\phi, \quad y = a\cosh\xi\cos\eta\sin\phi, \quad z = a\sinh\xi\sin\eta.$$

- Scale factors [gex75]: $h_\xi = h_\eta = a\sqrt{\sinh^2\xi + \sin^2\eta}, \quad h_\phi = a\cosh\xi\cos\eta$.
- Mutually orthogonal 2D manifolds [gex75]:
  - ▷ $\phi =$ const: planes containing the z-axis.
  - ▷ $\xi =$ const: ellipsoidal surfaces of rotation (red).
  - ▷ $\eta =$ const: hyperboloidal surfaces of rotation (blue).



Blue lines: $\eta = n\pi/8, \quad n = 0, 1, \ldots, 8$ (from right to left).
Red lines: $\xi = 0.5, 1, 0, 1, 5, 2.0, 2.25$ (from inside out).
The z-axis of rotation is vertical

Ellipsoidal coordinates:

Ellipsoids, in general, have three different principal semi-axes $a > b > c > 0$. If two semi-axes are equal ($a = b$ or $b = c$), we have a spheroid (ellipsoid of rotation). If all three semi-axes are equal ($a = b = c$), we have a sphere.

- Curvilinear coordinates: $\lambda,\ \mu,\ \nu$.
- Range of λ, μ, ν depends on the values of a^2, b^2, c^2 in a complicated way.
- Transformation relations to Cartesian coordinates: The three relations on the left have the solutions stated on the right [gex76].

$$\frac{x^2}{a^2-\lambda} + \frac{y^2}{b^2-\lambda} + \frac{z^2}{c^2-\lambda} = 1, \quad | \quad x = \pm\sqrt{\frac{(a^2-\lambda)(a^2-\mu)(a^2-\nu)}{(a^2-b^2)(a^2-c^2)}},$$

$$\frac{x^2}{a^2-\mu} + \frac{y^2}{b^2-\mu} + \frac{z^2}{c^2-\mu} = 1, \quad | \quad y = \pm\sqrt{\frac{(b^2-\lambda)(b^2-\mu)(b^2-\nu)}{(b^2-a^2)(b^2-c^2)}},$$

$$\frac{x^2}{a^2-\nu} + \frac{y^2}{b^2-\nu} + \frac{z^2}{c^2-\nu} = 1. \quad | \quad z = \pm\sqrt{\frac{(c^2-\lambda)(c^2-\mu)(c^2-\nu)}{(c^2-a^2)(c^2-b^2)}}.$$

- Scale factors [gex76]:

$$h_\lambda = \frac{1}{2}\sqrt{\frac{(\mu-\lambda)(\nu-\lambda)}{(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)}},$$

$$h_\mu = \frac{1}{2}\sqrt{\frac{(\nu-\mu)(\lambda-\mu)}{(a^2-\mu)(b^2-\mu)(c^2-\mu)}},$$

$$h_\nu = \frac{1}{2}\sqrt{\frac{(\lambda-\nu)(\mu-\nu)}{(a^2-\nu)(b^2-\nu)(c^2-\nu)}}.$$

- The mutually orthogonal 2D manifolds are complicated objects. Here explore special cross sections of such manifolds.

  Consider the case with $a^2 = 3, \quad b^2 = 2, \quad c^2 = 1$.

  #1 Set $\mu = 2$, which implies $y = 0$.
  Ranges $-\infty < \lambda \leq 1$ and $1 \leq \nu \leq 3$ then cover the xz-plane.
  Keeping ν fixed produces (blue) hyperbolas and keeping λ fixed (red) ellipses in the graph on the left.

#2 Set $\nu = 1$, which implies $z = 0$.
Ranges $-\infty < \lambda \leq 2$ and $2 \leq \mu \leq 3$ then cover the xy-plane.
Keeping μ fixed produces (blue) hyperbolas and keeping λ fixed (red) ellipses in the graph in the middle.

#3 Set $\lambda = 3$, which implies $x = 0$.
Ranges $-\infty < \nu \leq 1$ and $1 \leq \mu \leq 2$ then cover the yz-plane.
Keeping μ fixed produces (blue) hyperbolas and keeping ν fixed (red) ellipses in the graph on the right.