Coordinate Systems I [gmd2-A]

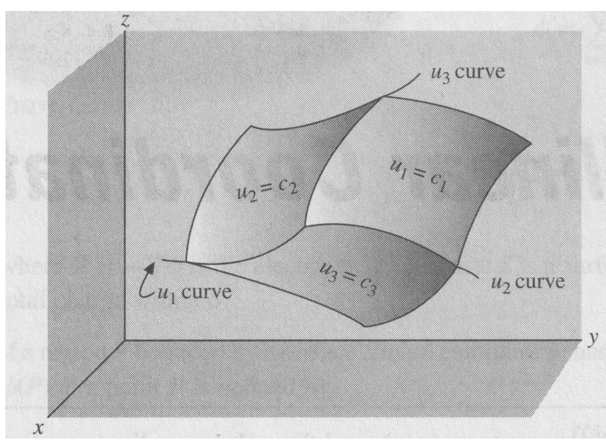
General coordinates:

Points in three-dimensional (3D) Euclidean space are specified by 3 coordinates. The default coordinates x_1, x_2, x_3 (x, y, z) are rectangular (Cartesian).

General coordinates u_1, u_2, u_3 are curvilinear and non-orthogonal. Coordinate transformations are assumed to be single-valued (excepts for isolated points of singularity) and described by differentiable functional relations,

$$x_i(u_1, u_2, u_3), \quad u_i(x_1, x_2, x_3) \quad : \ i = 1, 2, 3.$$

- Coordinate surface: Keeping one coordinate fixed, $u_i = c_i$, identifies a 2D manifold (the u_i-surface).
- Coordinate curve: Keeping two coordinates fixed, $u_i = c_i$ and $u_j = c_j$, identifies a 1D manifold (the u_k-curve).
- Coordinate point: Assigning values to all three coordinates identifes a 0D manifold. Here three coordinate curves and three coordinate surfaces meet.



[image from Spiegel et al. 2009]

In general, the three coordinate surfaces are not oriented perpendicular to each other at points of intersection (coordinate curves). This is the case for orthogonal coordinate systems only.

In the rectangular coordinate system, the coordinate curves are mutually orthogonal straight lines and the coordinate surfaces are mutually orthogonal planes.

Position vector: $\mathbf{r} = x_1\hat{\mathbf{i}} + x_2\hat{\mathbf{j}} + x_3\hat{\mathbf{k}} = \mathbf{r}(u_1, u_2, u_3)$.

Tangent vectors: $\dfrac{\partial \mathbf{r}}{\partial u_i} = \left|\dfrac{\partial \mathbf{r}}{\partial u_i}\right| \mathbf{e}_i = h_i\,\mathbf{e}_i$.

Scale factors: $h_i = \left|\dfrac{\partial \mathbf{r}}{\partial u_i}\right|$.

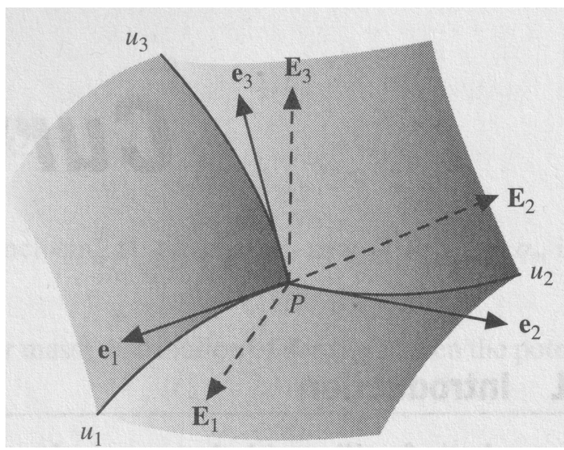
Unit tangent vectors: $\mathbf{e}_i$ (tangent to coordinate curves).

Coordinate surfaces: $u_i(x_1, x_2, x_3)$.

Normal vectors: $\nabla u_i = |\nabla u_i|\mathbf{E}_i$.

Unit normal vectors: $\mathbf{E}_i$ (normal to coordinate surfaces).

Note the dual role of the u_i as coordinates and as coordinate surfaces.



[image from Spiegel et al. 2009]

In orthogonal coordinate systems, the two sets of unit vectors coincide.

A vector $\mathbf{A}$ can be expanded using alternative sets of basis vectors:[1]

- Contravariant expansion: $\mathbf{A} = \sum_i C_i \dfrac{\partial \mathbf{r}}{\partial u_i} = \sum_i \bar{C}_i\,\mathbf{e}_i$.
- Covariant expansion: $\mathbf{A} = \sum_i c_i \nabla u_i = \sum_i \bar{c}_i\,\mathbf{E}_i$.

[1]The precise meaning of the terms 'contravariant' and 'covariant' becomes will be explained in the context of tensor analysis [gmd5].

Displacement in contravariant expansion:

$$d\mathbf{s} = dx_1\hat{\mathbf{i}} + dx_2\hat{\mathbf{j}} + dx_3\hat{\mathbf{k}} = \sum_j \left[\frac{\partial x_1}{\partial u_j}\, du_j\hat{\mathbf{i}} + \frac{\partial x_2}{\partial u_j}\, du_j\hat{\mathbf{j}} + \frac{\partial x_3}{\partial u_j}\, du_j\hat{\mathbf{k}}\right]$$

$$= \sum_j \frac{\partial \mathbf{s}}{\partial u_j} du_j = \sum_j \left|\frac{\partial \mathbf{s}}{\partial u_j}\right| du_j\mathbf{e}_j = \sum_j h_j du_j \mathbf{e}_j.$$

Displacement components: $dx_i = \sum_j \frac{\partial x_i}{\partial u_j}\, du_j$.

Squared arc length:

$$ds^2 = d\mathbf{s}\cdot d\mathbf{s} = \sum_i dx_i^2 = \sum_{ijk} \frac{\partial x_i}{\partial u_j}\frac{\partial x_i}{\partial u_k}\, du_j du_k = \sum_{jk} g_{jk} du_j du_k.$$

Metric coefficients: $g_{jk} = \sum_i \frac{\partial x_i}{\partial u_j}\frac{\partial x_i}{\partial u_k}$.

Coordinate systems of this generality will be further discussed elsewhere.

In the following, we focus on orthogonal coordinates. Here the $\mathbf{e}_i$ are an orthonormal set. We can write

$$ds^2 = \sum_j h_j^2 du_j^2, \quad g_{jk} = h_j^2 \delta_{jk}, \quad h_j^2 = \sum_i \left(\frac{\partial x_i}{\partial u_j}\right)^2.$$

The displacement increments in the three mutually orthogonal directions at any given point become

$$ds_j = h_j du_j.$$

The du_i are not necessarily all length elements. They can be angular increments, for example.

However, ds_1, ds_2, ds_3 are length elements, $ds_1ds_2, ds_2ds_3, ds_3ds_1$ are area elements, and $ds_1ds_2ds_3$ is a volume element.

Orthogonal coordinates:

If the three 2D manifolds associated with a set of coordinates in 3D space are oriented perpendicular to each other at every point, we have an orthogonal coordinate system.

The three most important orthogonal coordinate systems are the following:[2]

- ▷ *Cartesian (rectilinear) coordinates*: $x,\ y,\ z$.
  The manifolds are the (mutually perpendicular) xy, xz, and yz planes.
- ▷ *Cylindrical coordinates*: $\rho,\ \phi,\ z$.
  The three manifolds are a cylindrical surface, centered at the z axis, and two planes, one oriented perpendicular to the z axis and the other containing the z axis.
- ▷ *Spherical coordinates*: $r,\ \theta,\ \phi$.
  The three manifolds are a spherical surface, centered the origin, a cone about the z axis, and a plane containing the z axis.

The three sets of coordinates are tailor-made for situations with planar, cylindrical, and spherical symmetry, respectively. They remain useful if these symmetries are broken in particular ways.

Associated with each triplet of coordinates are triplets of mutually perpendicular unit vectors $\mathbf{e}_i$ (right-handed triads). In general, these triplets of unit vectors change their orientation from point to point in space.

Cylindrical and spherical coordinates are *curvilinear*. The triplet of unit vectors have variable orientations. Cartesian coordinates are *rectilinear*. The orientation of the triplet of unit vectors is fixed.

Rectilinear coordinates share all attributes of curvilinear coordinates, but embody simplifying features.

[2] The radial coordinate has a different meaning in the cylindrical and spherical sets. It is often named ρ in the former set. The azimuthal angle ϕ is common to cylindrical and spherical coordinates, whereas the polar angle θ is special to the latter.

Gradient, divergence, curl, and Laplacian:

The notation employed here is applicable, in equal measure, to Cartesian, cylindrical, and spherical coordinates in 3D Euclidean space. The basic ingredients are

- ▷ $\mathbf{e}_i$: mutually orthogonal unit vectors,
- ▷ h_i: scale factors,
- ▷ du_i: coordinate increments.

Infinitesimal displacement: $d\mathbf{s} = h_1 du_1 \mathbf{e}_1 + h_2 du_2 \mathbf{e}_2 + h_3 du_3 \mathbf{e}_3$.

– Gradient of a scalar field from its differential:

$$df = \nabla f \cdot d\mathbf{s} = \sum_i (\nabla f)_i h_i du_i \doteq \sum_i \frac{\partial f}{\partial u_i} du_i$$

$$\Rightarrow \ (\nabla f)_i = \frac{1}{h_i}\frac{\partial f}{\partial u_i} \quad \Rightarrow \ \nabla f = \sum_i \frac{1}{h_i}\frac{\partial f}{\partial u_i}\,\mathbf{e}_i.$$

– Divergence of a vector field with use of Gauss's theorem:[3]

$$(\nabla \cdot \mathbf{F}) dV = \mathbf{F} \cdot d\mathbf{a}.$$

Element of volume: $dV = h_1 du_1 h_2 du_2 h_3 du_3$.
Elements of areas of enclosing surface: $da_i = h_j du_j h_k du_k$.

$$\mathbf{F} \cdot d\mathbf{a} = d(F_1 h_2 h_3) du_2 du_3 + d(F_2 h_1 h_3) du_1 du_3 + d(F_3 h_1 h_2) du_1 du_2$$

$$\Rightarrow \ \frac{\mathbf{F}\cdot d\mathbf{a}}{dV} = \frac{1}{h_1 h_2 h_3}\left[\frac{d(F_1 h_2 h_3)}{du_3} + \frac{d(F_2 h_1 h_3)}{du_2} + \frac{d(F_3 h_1 h_2)}{du_3}\right]$$

$$\Rightarrow \ \nabla \cdot \mathbf{F} = \frac{1}{h_1 h_2 h_3}\sum_{\{ijk\}} \frac{\partial}{\partial u_i}(F_i h_j h_k)$$

$$= \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(F_1 h_2 h_3) + \frac{\partial}{\partial u_2}(F_2 h_3 h_1) + \frac{\partial}{\partial u_3}(F_3 h_1 h_2)\right].$$

[3]Here and in the following we invoke sums over cyclically permuted indices using the notation, $\{ijk\} = \mathrm{cycl}\{123\} \doteq \{123, 231, 312\}$.

– Curl of a vector function with use of Stokes' theorem:

$$(\nabla \times \mathbf{F})_k da_k = (\mathbf{F} \cdot d\mathbf{l})_k.$$

Element of area of open surface: $da_k = h_i du_i h_j du_j$.
Elements of loop surrounding loop: $dl_i = h_i du_i,\ dl_j = h_j du_j$.

$$\begin{aligned}
(\nabla \times \mathbf{F})_k h_i du_i h_j du_j &= d(F_j h_j) du_j - d(F_i h_i) du_i \\
\Rightarrow \ \frac{(\mathbf{F} \cdot d\mathbf{l})_k}{da_k} &= \frac{1}{h_i h_j}\left[\frac{d(F_j h_j)}{du_i} - \frac{d(F_i h_i)}{du_j}\right] \\
\Rightarrow \ (\nabla \times \mathbf{F})_k &= \frac{1}{h_i h_j}\left[\frac{\partial}{\partial u_i}(F_j h_j) - \frac{\partial}{\partial u_j}(F_i h_i)\right] \\
\Rightarrow \ \nabla \times \mathbf{F} &= \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1 \hat{\mathbf{e}}_1 & h_2 \hat{\mathbf{e}}_2 & h_3 \hat{\mathbf{e}}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix}.
\end{aligned}$$

– Laplacian from the divergence of the gradient of a scalar function:

$$\begin{aligned}
&\nabla^2 f = \nabla \cdot (\nabla f) = \frac{1}{h_1 h_2 h_3} \sum_{\{ijk\}} \frac{\partial}{\partial u_i}\left(\frac{h_j h_k}{h_i}\frac{\partial f}{\partial u_i}\right) \\
&= \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial f}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial f}{\partial u_3}\right)\right].
\end{aligned}$$

The last expression uses the results of the first two items.

Determination of scale factors:

The unit scale factors $h_x = h_y = h_z = 1$ are a key simplifying feature of Cartesian coordinates $(x, y, z) \doteq (x_1, x_2, x_3)$.

Scale factors for (orthogonal) curvilinear coordinates (u_1, u_2, u_3) can then be determined from their transformation relations $x_i(u_1, u_2, u_3)$ as follows:

$$\frac{\partial \mathbf{s}}{\partial u_j} = \frac{\partial x_1}{\partial u_j}\,\hat{\mathbf{i}} + \frac{\partial x_2}{\partial u_j}\,\hat{\mathbf{j}} + \frac{\partial x_3}{\partial u_j}\,\hat{\mathbf{k}} = h_j \mathbf{e}_j.$$

$$\Rightarrow \; h_j = \left|\frac{\partial \mathbf{s}}{\partial u_j}\right| = \sqrt{\left(\frac{\partial x_1}{\partial u_j}\right)^2 + \left(\frac{\partial x_2}{\partial u_j}\right)^2 + \left(\frac{\partial x_3}{\partial u_j}\right)^2}.$$

Orthogonal unit vectors:

$$\mathbf{e}_1, \quad \mathbf{e}_2, \quad \mathbf{e}_3 \quad \Rightarrow \; \mathbf{e}_j \cdot \mathbf{e}_k = \delta_{jk}.$$

Application to cylindrical coordinates:

▷ Curvilinear coordinates:

$$u_1 = \rho, \quad u_2 = \phi, \quad u_3 = z.$$

▷ Transformation relations:

$$x_1 = \rho\cos\phi, \quad x_2 = \rho\sin\phi, \quad x_3 = z.$$

▷ $\dfrac{\partial x_1}{\partial \rho} = \cos\phi, \quad \dfrac{\partial x_2}{\partial \rho} = \sin\phi, \quad \dfrac{\partial x_3}{\partial \rho} = 0.$

$$\Rightarrow \; h_\rho = \sqrt{\cos^2\phi + \sin^2\phi} = 1.$$

▷ $\dfrac{\partial x_1}{\partial \phi} = -\rho\sin\phi, \quad \dfrac{\partial x_2}{\partial \phi} = \rho\cos\phi, \quad \dfrac{\partial x_3}{\partial \phi} = 0.$

$$\Rightarrow \; h_\phi = \sqrt{\rho^2\sin^2\phi + \rho^2\cos^2\phi} = \rho.$$

▷ $\dfrac{\partial x_1}{\partial z} = 0, \quad \dfrac{\partial x_2}{\partial z} = 0, \quad \dfrac{\partial x_3}{\partial z} = 1.$

$$\Rightarrow \; h_z = 1.$$

Application to spherical coordinates:

▷ Curvilinear coordinates:

$$u_1 = r, \quad u_2 = \theta, \quad u_3 = \phi.$$

▷ Transformation relations:

$$x_1 = \sin\theta\cos\phi, \quad x_2 = r\sin\theta\sin\phi, \quad x_3 = r\cos\theta.$$

▷ $\dfrac{\partial x_1}{\partial r} = \sin\theta\cos\phi, \quad \dfrac{\partial x_2}{\partial r} = \sin\theta\sin\phi, \quad \dfrac{\partial x_3}{\partial r} = \cos\theta.$

$$\Rightarrow \; h_r = \sqrt{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta} = 1.$$

▷ $\dfrac{\partial x_1}{\partial \theta} = r\cos\theta\cos\phi, \quad \dfrac{\partial x_2}{\partial \theta} = r\cos\theta\sin\phi, \quad \dfrac{\partial x_3}{\partial \theta} = -r\sin\theta.$

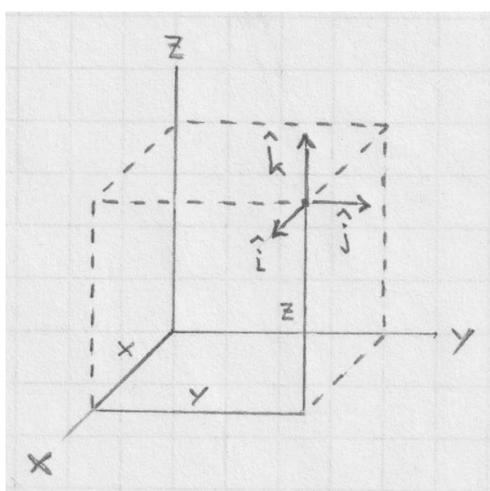
$$\Rightarrow \; h_\theta = \sqrt{r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta} = r.$$

▷ $\dfrac{\partial x_1}{\partial \phi} = -r\sin\theta\sin\phi, \quad \dfrac{\partial x_2}{\partial \phi} = r\sin\theta\cos\phi, \quad \dfrac{\partial x_3}{\partial \phi} = 0.$

$$\Rightarrow \; h_z = \sqrt{r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi} = r\sin\theta.$$

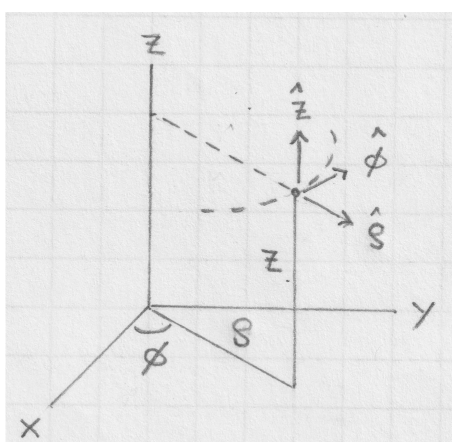
Cartesian coordinates:

position	$\mathbf{x} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$
displacement	$d\mathbf{s} = dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}} + dz\,\hat{\mathbf{k}}$
range	$-\infty < x, y, z < \infty$
scale factors	$h_x = 1,\ h_y = 1,\ h_z = 1$
volume element	$dV = dx\,dy\,dz$
area elements	$dA_x = dy\,dz,\ dA_y = dx\,dz,\ dA_z = dx\,dy$
line elements	$ds_x = dx,\ ds_y = dy,\ ds_z = dz$
gradient	$\nabla f = \frac{\partial f}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\,\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\,\hat{\mathbf{k}}$
divergence	$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$
Laplacian	$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$
curl	$\nabla \times \mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{\mathbf{k}}$



Cylindrical coordinates:

position	$\mathbf{x} = \rho\,\hat{\boldsymbol{\rho}} + z\,\hat{\mathbf{z}}$
displacement	$d\mathbf{s} = d\rho\,\hat{\boldsymbol{\rho}} + \rho\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$
range	$\rho \geq 0, \quad 0 \leq \phi < 2\pi, \quad -\infty < z < \infty$
transformation	$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = z$
scale factors	$h_\rho = 1,\ h_\phi = \rho,\ h_z = 1$
volume element	$dV = \rho d\rho\,d\phi\,dz$
area elements	$dA_\rho = \rho d\phi\,dz,\ dA_\phi = d\rho\,dz,\ dA_z = \rho d\rho\,d\phi$
line elements	$ds_\rho = d\rho,\ ds_\phi = \rho\,d\phi,\ ds_z = dz$
gradient	$\nabla f = \frac{\partial f}{\partial \rho}\,\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\,\hat{\boldsymbol{\phi}} + \frac{\partial f}{\partial z}\,\hat{\mathbf{z}}$
divergence	$\nabla \cdot \mathbf{F} = \frac{1}{\rho}\frac{\partial(\rho F_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}$
Laplacian	$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\,\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$
curl	$\nabla \times \mathbf{F} = \left(\frac{1}{\rho}\frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z}\right)\hat{\boldsymbol{\rho}} + \left(\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho}\right)\hat{\boldsymbol{\phi}} + \frac{1}{\rho}\left(\frac{\partial(\rho F_\phi)}{\partial \rho} - \frac{\partial F_\rho}{\partial \phi}\right)\hat{\mathbf{z}}$



Spherical coordinates:

position	$\mathbf{x} = r\,\hat{\mathbf{r}}$
displacement	$d\mathbf{s} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$
range	$r \geq 0, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi < 2\pi$
transformation	$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$
scale factors	$h_r = 1,\ h_\theta = r,\ h_\phi = r\sin\theta$
volume element	$dV = r^2 dr\sin\theta\,d\theta\,d\phi$
area elements	$dA_r = r^2\sin\theta\,d\theta\,d\phi,\ dA_\theta = r dr\sin\theta\,d\phi,\ dA_\phi = r dr\,d\theta,$
line elements	$ds_r = dr,\ ds_\theta = r\,d\theta,\ ds_\phi = r\sin\theta\,d\phi$
gradient	$\nabla f = \frac{\partial f}{\partial r}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\,\hat{\boldsymbol{\phi}}$
divergence	$\nabla\cdot\mathbf{F} = \frac{1}{r^2}\frac{\partial(r^2F_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial\sin\theta F_\theta}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}$
Laplacian	$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$
curl	$\nabla\times\mathbf{F} = \frac{1}{r\sin\theta}\left(\frac{\partial(\sin\theta\,F_\phi)}{\partial\theta} - \frac{\partial F_\theta}{\partial\phi}\right)\hat{\mathbf{r}}$ $+\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\phi} - \frac{\partial(r\,F_\phi)}{\partial r}\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\frac{\partial(rF_\theta)}{\partial r} - \frac{\partial F_r}{\partial\theta}\right)\hat{\boldsymbol{\phi}}$