Partial Differential Equations I [gmd11-A]

A partial differential equation (PDE) involves derivatives with respect to more than one independent variable of a multi-variable function.

Common notation in PDEs for partial derivatives of a function $u(x, y, z, t)$:

$$u_x \doteq \frac{\partial u}{\partial x}, \quad u_t \doteq \frac{\partial u}{\partial t}, \quad u_{xx} \doteq \frac{\partial^2 u}{\partial x^2}, \quad u_{xy} \doteq \frac{\partial^2 u}{\partial x \partial y}, \quad \ldots \tag{1}$$

Classification:

PDEs are primarily classified according to three criteria: (i) order, (ii) linearity, and (iii) homogeneity. Criterion (iii) only applies if (ii) is satisfied.

One important secondary classification applies to linear, second-order PDEs for functions of two variables such as $u(x, y)$:

$$\underbrace{au_{xx} + bu_{xy} + cu_{yy}}_{\text{principal part}} + du_x + eu_y + fu = g, \tag{2}$$

where a, b, c, d, e, f, g are either constants or functions of x, y.

The nature of solutions is dominated by the principal part of the PDE, specifically by the sign of the discriminant,

$$b^2 - 4ac. \tag{3}$$

Three types of solutions:

- ▷ $b^2 - 4ac > 0$: hyperbolic,
- ▷ $b^2 - 4ac = 0$: parabolic,
- ▷ $b^2 - 4ac < 0$: elliptic.

If a, b, c are not all constant, then the type of solution for the same PDE may vary from region to region.

Well-known PDEs of each type are the following:

- the wave equation, $u_{tt} = c^2 u_{xx}$, is hyperbolic,
- the diffusion equation, $u_t = Du_{xx}$, is parabolic,
- the Laplace equation, $u_{xx} + u_{yy} = 0$, is elliptic.

Structure of general solution according to type:

A simple demonstration is possible for a PDE with principal part only and constant parameters a, b, c:

$$au_{xx} + bu_{xy} + cu_{yy} = 0, \quad a \neq 0. \tag{4}$$

The goal is to explore the structure of the general solution (unrestricted by subsidiary conditions).

Step #1: Linear variable transformation:

- ▷ $u(x, y) = \bar{u}(r, s)$ with $r = \alpha x + \beta y$ and $s = \gamma x + \delta y$.
- ▷ $u_x = \alpha\bar{u}_r + \gamma\bar{u}_s, \quad u_y = \beta\bar{u}_r + \delta\bar{u}_s$.
- ▷ $u_{xx} = \alpha^2\bar{u}_{rr} + 2\alpha\gamma\bar{u}_{rs} + \gamma^2\bar{u}_{ss}, \quad u_{yy} = \beta^2\bar{u}_{rr} + 2\beta\delta\bar{u}_{rs} + \delta^2\bar{u}_{ss}$,
  $u_{xy} = \alpha\beta\bar{u}_{rr} + (\alpha\delta + \beta\gamma)\bar{u}_{rs} + \gamma\delta\bar{u}_{ss}$.
- ▷ Transformed PDE:

$$\begin{aligned}\left(a\alpha^2 + b\alpha\beta + c\beta^2\right)\bar{u}_{rr} &+ \left(a\gamma^2 + b\gamma\delta + c\delta^2\right)\bar{u}_{ss} \\ &+ \left[2a\alpha\gamma + b(\alpha\delta + \beta\gamma) + 2c\beta\delta\right]\bar{u}_{rs} = 0.\end{aligned} \tag{5}$$

Step#2: Simplify (5) by special choice of $\alpha, \beta, \gamma, \delta$:

- ▷ Set $\beta = \delta = 1$.
- ▷ Choose α, γ such that coefficients of $\bar{u}_{rr}$ and $\bar{u}_{ss}$ in (5) vanish.

$$\begin{aligned}&a\alpha^2 + b\alpha + c = 0, \quad a\gamma^2 + b\gamma + c = 0, \\ &\alpha = \frac{1}{2a}\left[-b + \sqrt{b^2 - 4ac}\right], \quad \gamma = \frac{1}{2a}\left[-b - \sqrt{b^2 - 4ac}\right].\end{aligned} \tag{6}$$

- ▷ Simplified PDE:

$$\begin{aligned}a\alpha^2 + b\alpha\beta + c\beta^2 = a\gamma^2 + b\gamma\delta + c\delta^2 = 0, \\ 2a\alpha\gamma + b(\alpha\delta + \beta\gamma) + 2c\beta\delta = \frac{4ac - b^2}{a}.\end{aligned}$$

$$\Rightarrow \frac{4ac - b^2}{a}\bar{u}_{rs} = 0. \tag{7}$$

Step#3: Examine general solution for transformed variables:

- ▷ Case of non-degenerate solutions (6): $b^2 \neq 4ac$.

$$\bar{u}_{rs} = 0 \quad \Rightarrow \bar{u}(r, s) = F(r) + G(s). \tag{8}$$

▷ Case of degenerate solutions (6): $b^2 = 4ac$.

Modified transformation: $\alpha = -b/2a,\ \beta = \gamma = 1,\ \delta = 0$

$$\Rightarrow\ a\alpha^2 + b\alpha\beta + c\beta^2 = 2a\alpha\gamma + b(\alpha\delta + \beta\gamma) + 2c\beta\delta = 0,$$
$$a\gamma^2 + b\gamma\delta + c\delta^2 = a \neq 0$$
$$\Rightarrow\ a\bar{u}_{ss} = 0.$$

$$\bar{u}_{ss} = 0 \quad \Rightarrow\ \bar{u}(r,s) = F(r) + sG(s). \tag{9}$$

Step#4: Return to original variables:

▷ Hyperbolic case ($b^2 > 4ac$):

$$u(x,y) = F(\alpha x + y) + G(\gamma x + y), \quad \alpha, \gamma \in \mathbb{R}. \tag{10}$$

▷ Elliptic case ($b^2 < 4ac$):

$$u(x,y) = F(\alpha x + y) + G(\gamma x + y), \quad \alpha = \gamma^* \in \mathbb{C}. \tag{11}$$

▷ Parabolic case ($b^2 = 4ac$):

$$u(x,y) = F(\alpha x + y) + xG(\alpha x), \quad \alpha \in \mathbb{R}. \tag{12}$$

Hyperbolic case applied to wave equation, $u_{tt} = v^2 u_{xx}$:

▷ Traveling-wave solution: $a = 1,\ b = 0,\ c = -\dfrac{1}{v^2} \quad \Rightarrow\ \alpha = \dfrac{1}{v},\ \gamma = -\dfrac{1}{v}.$

$$\Rightarrow\ u(x,t) = F(x - vt) + F(x + vt). \tag{13}$$

Elliptic solution applied Laplace equation, $u_{xx} + u_{yy} = 0$:

▷ Conjugate-functions solution: $a = c = 1,\ b = 0 \quad \Rightarrow\ \alpha = \imath,\ \gamma = -\imath.$

$$\Rightarrow\ u(x,t) = F(y + \imath x) + F(y - \imath x). \tag{14}$$

Subsidiary conditions:

Whereas the general solution of an ODE contains arbitrary constants, the general solution of a PDE contains arbitray functions.

Subsidiary conditions fall into two categories:

– Boundary conditions involve a function h_S given at the surface S of the region Ω in which a solution is sought.

  ▷ Dirichlet: $u(x, y) = h_S$ on S,

  ▷ Neumann: $\partial u/\partial n = h_S$ on S,

  ▷ Robin: $\alpha u + \beta(\partial u/\partial n) = h_S$ on S.

– Initial conditions involve a function h_Ω given at $t = 0$ throughout the region Ω in which a solution is sought.

  ▷ Cauchy: $u = h_\Omega$ in Ω at $t = 0$.

The choice of subsidiary conditions is more delicate for PDEs than for ODEs. Well-posed PDE problems must satisfy three (nontrivial) conditions:

▷ a solution must exist,

▷ the solution must be unique,

▷ the solution must depend continuously on parameters of the PDE and on the subsidiary conditions.

A PDE problem is ill-posed if not all three conditions are met. Overspecified problems tend to have no solution, whereas the solution of underspecified problems tends to be not unique.