Ordinary Differential Equations I [gmd10-A]

An *ordinary differential equation* (ODE) has terms involving a function of one variable and derivatives of that function with respect to the independent variable. ODEs of order n include derivatives of order up to n.

If the independent variable is spatial [temporal] in nature, the common shorthand notation for derivatives is

$$\frac{dy}{dx} \doteq y', \quad \frac{d^2y}{dx^2} \doteq y'', \quad \frac{d^ny}{dx^n} \doteq y^{(n)}, \qquad \left[\frac{dy}{dt} \doteq \dot{y}, \quad \frac{d^2y}{dt^2} \doteq \ddot{y}, \quad \frac{d^ny}{dt^n} \doteq y^{(n)}\right].$$

Categories of solutions:

Solutions of an n^{th}-order ODE are categorized as follows:

- The *general solution* contains n integration constants, representing an n-parameter family of curves. Conversely, an n-parameter family of curves can be shown to be the general solution of an n^{th}-order ODE.
- A *particular solution* (single curve) assigns values to the integration constants on account of subsidiary conditions.
- For some ODEs (most commonly of 1^{st}-order) a *singular solution* (also a single curve) exists which cannot be inferred from the general solution by way of specifying integration constants.

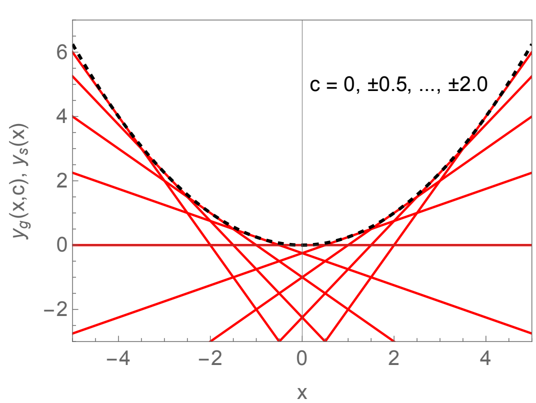
Example:

▷ Nonlinear 1^{st}-order ODE: $y'(y'-x)+y=0$.

▷ General solution: $y_{\text{g}}(x)=c(x-c)$.

▷ Particular solutions are linear functions for specific values of c.

▷ The singular solution, $y_{\text{s}}(x)=\frac{1}{4}x^2$, is tangential to each particular solution at $x=2c$.



The general solution of an n^{th}-order ODE requires n subsidiary conditions to make it a particular solution.

- Initial conditions involve only one value of the independent variable.
- Boundary conditions involve at least two values of the independent variable.

Linear ODEs of any order are amenable to special methods of wide scope including integral transforms. They will be discussed separately [gam8].

First-order ODEs:

Standard form: $\dfrac{dy}{dx} = f(x, y)$.

Differential form: $g(x, y)dx + h(x, y)dy = 0$.

Relation between the two forms: $f(x, y) = -\dfrac{g(x, y)}{h(x, y)}$.

The differential version is not unique for an ODE given in standard form.

One-parameter general solution: $y(x, c)$.

Particular solutions are curves that cannot intersect themselves. At each point, the slope is unique.

Special cases are solvable by elementary means as described in the following:

Separation of variables:

Differential form of ODE has factorizing coefficients:

$$g_1(x)g_2(y)dx + h_1(x)h_2(y)dy = 0.$$

Differentials of x and y can be separated and integrated independently:

$$\int dx\, \frac{g_1(x)}{h_1(x)} + \int dy\, \frac{h_2(y)}{g_2(y)} = c \quad \Rightarrow\; F_x(x) + F_y(y) = a.$$

A one-parameter general solution $y(x, a)$ is implicit in the last relation. It may be a single-valued or multiple-valued function.

Relevant exercises: [gex4], [gex5], [gex6].

Exact differentials:

Any 1st-order ODE given in standard form is expressible in differential form:

$$g(x,y)dx + h(x,y)dy = 0.$$

General form of exact differential : $dU = \dfrac{\partial U}{\partial x}\,dx + \dfrac{\partial U}{\partial y}\,dy.$

Equality of second cross derivatives: $\dfrac{\partial^2 U}{\partial x \partial y} = \dfrac{\partial^2 U}{\partial y \partial x}.$

Condition for ODE in differential form to represent an exact differential:

$$g(x,y) = \frac{\partial U}{\partial x}, \quad h(x,y) = \frac{\partial U}{\partial y} \qquad \Rightarrow \quad \frac{\partial g}{\partial y} = \frac{\partial h}{\partial x}.$$

The integral of an exact differential, $dU = 0$, is path-independent, but must be evaluated along a specific path C from a reference point of choice to (x, y):

$$\int_C dU = U(x,y) = c.$$

The result of the integral implies a one-parameter relation between y and x. It may represent a single-valued or multiple-valued function $y(x, a)$.

Relevant exercises: [gex7].

Integrating factor:

If the ODE in differential form,

$$g(x,y)dx + h(x,y)dy = 0,$$

does not represent an exact differential, a common factor $m(x, y)$ in both terms may make it an exact differential:

$$\frac{\partial(mg)}{\partial y} = \frac{\partial(mh)}{\partial x} \;\Rightarrow\; mgdx + mhdy = dU(x,y) = 0 \;\Rightarrow\; U(x,y) = c.$$

A single-valued or multiple-valued one-parameter general solution $y(x, a)$ is then again implicit. The integrating factor is, in general, not unique.

Relevant exercises: [gex12].

Linearity:

Among the many ways linear ODEs can be solved, we describe here a trick applicable to 1$^{\text{st}}$-order linear ODEs which is akin to an integrating factor.

The ODE to be solved is given in the form,

$$\frac{dy}{dx} + P(x)y = Q(x).$$

Integrating factor constructed from the given coefficient $P(x)$:

$$m(x) \doteq \exp\left(\int dx\, P(x)\right).$$

Equivalent ODE:

$$\frac{d}{dx}\Big(m(x)y(x)\Big) = m(x)Q(x).$$

Equivalence demonstrated by carrying out derivative:

$$m(x)\frac{dy}{dx} + m(x)P(x)y(x) = m(x)Q(x).$$

Solution of equivalent ODE via integration:

$$m(x)y(x) = \int_{x_0}^{x} dx'\, m(x')Q(x') \quad \Rightarrow \; y(x) = \frac{1}{m(x)}\int_{x_0}^{x} dx'\, m(x')Q(x').$$

Relevant exercises: [gex13].

Homogeneity:[1]

If the function $f(x,y)$ in the standard form of a 1$^{\text{st}}$-order ODE scales as $f(tx,ty) = f(x,y)$, we can set $t = 1/x$ and write $f(x,y) = F(y/x)$.

ODE to be solved: $\dfrac{dy}{dx} = F(y/x)$.

Introduce auxiliary functions: $v(x) \doteq \dfrac{y(x)}{x}$.

$$\Rightarrow \; y(x) = v(x)x \quad \Rightarrow \; \frac{dy}{dx} = v(x) + x\frac{dv}{dx}.$$

[1]The term *homogeneous* in the context of ODEs has, in general, a different meaning, the one familiar from linear algebra.

Equivalent ODE for $v(x)$:

$$x\frac{dv}{dx} = F(v) - v.$$

Construct differential and separate variables:

$$xdv = [F(v) - v]dx \quad \Rightarrow \frac{dx}{x} = \frac{dv}{F(v) - v}.$$

Integration yields implicit expression for $v(x)$:

$$\ln x = \int \frac{dv}{F(v) - v} + c.$$

An implicit expression for the one-parameter solution $y(x, a)$ follows directly. It may be single valued or multiple-valued.

Relevant exercises: [gex14]

Bernoulli type:

An ODE is of the Bernoulli type if it can be cast in the form,

$$\frac{dy}{dx} + P(x)y = Q(x)y^n.$$

For $n = 0, 1$ we have linearity and proceed as described above.

For other values of n, we reduce the ODE to linearity by introducing the auxiliary function,

$$v(x) = [y(x)]^{1-n}.$$

The equivalent (linear) ODE for $v(x)$ is constructed as follows:

$$\begin{aligned}\frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx} &= (1-n)y^{-n}Q(x)y^n - (1-n)y^{-n}P(x)y \\ &= (1-n)Q(x) - (1-n)\underbrace{y^{1-n}}_{v}P(x).\end{aligned}$$

$$\Rightarrow \frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x).$$

The solution $y(x)$ of the Bernoulli-type ODE follows directly from the solution $v(x)$ of the linear ODE.

Relevant exercises: [gex15].

Convertibility:

A convertible ODE is expressible in the form,

$$y = g(x,p), \quad p \doteq \frac{dy}{dx}.$$

The conversion to an ODE for $p(x)$ may or may not bring a simplification. When it does, it is a useful move to make:

$$\frac{dy}{dx} = p = \frac{\partial g}{\partial x} + \frac{\partial g}{\partial p}\frac{dp}{dx} \quad \Rightarrow \quad \frac{dp}{dx} = \frac{p - \partial g/\partial x}{\partial g/\partial p} \doteq f(x,p).$$

The function $y(x)$ follows from $p(x)$ via integration. The integration constant must be chosen such that the original equation is satisfied.

Relevant exercises: [gex17].

Clairaut type:

If a convertible ODE is of the form,

$$y = xp + F(p), \quad p \doteq \frac{dy}{dx},$$

it is said to be of the Clairaut type. Its general (one-parameter) solution is the linear function,

$$y(x) = cx + F(c).$$

Note that the parameter c controls both the slope and the intercept of particular solutions.

There often exists a more interesting (singular) solution, which can be detected in graphical representations of particular solutions by systematic variation of the parameter c.

Relevant exercises: [gex25]

Special structures:

Various ODE structures lend themselves to serendipitous simplifications.

For example, consider the ODE: $\dfrac{dy}{dx} = F(\alpha x + \beta y)$.

Set $v = \alpha x + \beta y \quad \Rightarrow \quad \dfrac{dv}{dx} = \alpha + \beta\dfrac{dy}{dx} = \alpha + \beta F(v)$.

Separate variables and integrate: $\displaystyle\int \frac{dv}{\alpha + \beta F(v)} = \int dx + c.$

Second-order ODEs:

Second-order ODEs in standard form read

$$y''(x) = f\big(y'(x), y(x), x\big).$$

The general solution includes two parameters. Particular solutions, graphically represented by specific curves, satisfy two subsidiary conditions.

Solving 2$^{\text{nd}}$-order ODEs is, in general, quite challenging. The usual tricks developed for 1$^{\text{st}}$-order ODES are mostly inapplicable.

The solutions tend to have higher complexity, which is needed for the description of known physical phenomena. Some forms of complexity (deterministic chaos) were huge surprises at the time of discovery.

Coupled first-order ODEs:

Any 2$^{\text{nd}}$-order ODE can be expressed as a pair of coupled 1$^{\text{st}}$-order ODEs. We start from the 2$^{\text{nd}}$-order ODE in standard form and declare $y'(x)$ to be an independent function:

$$y''(x) = f\big(y'(x), y(x), x\big) \quad \Rightarrow \quad y'(x) = z(x), \quad z'(x) = f\big(z(x), y(x), x\big).$$

In the reverse direction, we start from the pair of 1$^{\text{st}}$-order ODEs,

$$y'(x) = f\big(y(x), z(x), x\big), \quad z'(x) = g\big(y(x), z(x), x\big),$$

Taking the derivative of the first ODE yields,

$$y''(x) = \frac{\partial f}{\partial y}\, y' + \frac{\partial f}{\partial z}\, z' + \frac{\partial f}{\partial x} = h\big(y(x), y'(x), x\big).$$

In the last step, we have substituted $z(x)$ from the inverted first ODE and $z'(x)$ from the second ODE.

The step from Lagrangian mechanics to Hamiltonian mechanics is associated with this switch in representation.

The Lagrangian $L(q,\dot{q})$ of a dynamical system with one degree of freedom can be transformed into its Hamiltonian, $H(q,p)$ via Legendre transform.

The Lagrange equation is a 2$^{\text{nd}}$-order ODE and the canonical equations a pair of 1$^{\text{st}}$-order ODEs:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}; \qquad \dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}.$$

The general structure of solutions of ODEs in the context dynamical systems of one degree of freedom is pursued in [gam3] including an outlook to more general dynamical systems.

Reduction to first-order ODE:

Reducing the order by one is a significant gain, which is possible under specific conditions including the following.

– The dependent variable is not explicitly present in the ODE [gex108]:

$$y''(x) = f\big(y'(x), x\big).$$

Set $y' = z$: $\Rightarrow$ $z'(x) = f\big(z(x), x\big)$. Then solve 1st-order ODE for $z(x)$, then integrate solution:

$$\Rightarrow \; y(x) = \int dx' z(x').$$

– The independent variable is not explicitly present in the ODE [gex109]:

$$y''(x) = f\big(y'(x), y(x)\big).$$

Set $y' = z$ and arrive at 1st-order ODE:

$$\Rightarrow \; y'' = \frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = z\,\frac{dz}{dy} \quad \Rightarrow \; z\,\frac{dz}{dy} = f(z, y).$$

Solve 1st-order ODE to arrive at $z(y)$. Then solve 1st-order ODE, $dy/dx = z(y)$, for $y(x)$.

Alternatively, use the inverse function $x(y)$ [gex116]:

$$y = G(x) \;\Leftrightarrow\; x = F(y)$$

$$\Rightarrow \; x' = \frac{dx}{dy} = F'(y), \quad x'' = F''(y)$$

$$\Rightarrow \; y' = \frac{dy}{dx} = G'(x) = \frac{1}{F'\big(G(x)\big)}$$

$$\Rightarrow \; y'' = G''(x) = -\frac{1}{\big[F'\big(G(x)\big)\big]^2} F''\big(G(x)\big)G'(x) = -\frac{F''\big(G(x)\big)}{\big[F'\big(G(x)\big)\big]^3}.$$

$$y'' = f\big(y', y\big) \quad \Rightarrow \; -\frac{x''}{(x')^3} = f(1/x', y).$$

Set $z(y) = x'(y)$:

$$\Rightarrow \; z' = -z^3 f(1/z, y).$$