Vector Analysis II [gmd1-B]

The main topic here is on integrals involving scalars and vectors along lines, across surfaces, over volumes bounded by surfaces, or simply over intervals of parameters.

Vector integrations:

The simplest kind of vector integrals involves vector functions:

$$\int du\,\mathbf{R}(u) = \hat{\mathbf{i}}\underbrace{\int du\,R_x(u)}_{S_x(u)+c_x} + \hat{\mathbf{j}}\underbrace{\int du\,R_y(u)}_{S_y(u)+c_y} + \hat{\mathbf{k}}\underbrace{\int du\,R_z(u)}_{S_z(u)+c_z} = \mathbf{S}(u) + \mathbf{c}.$$

$$\int_a^b du\,\mathbf{R}(u) = \mathbf{S}(b) - \mathbf{S}(a).$$

Spatial integrals of vector fields include *line integrals*, *surface integrals*, and *volume integrals* to be discussed separately.

Line integrals:

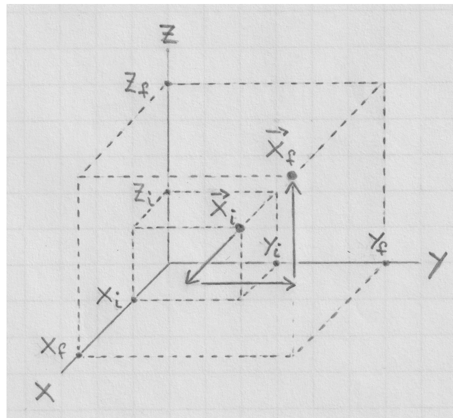
Integration along a path C in space is called a line integral. Line integrals along closed paths are named loop integrals.

▷ The line integral of a scalar field $f(\mathbf{x})$ yields a vector $\mathbf{I}$:

$$\begin{aligned}\mathbf{I} &= \int_C d\mathbf{x} f(\mathbf{x}) \\ &= \int_{x_i}^{x_f} dx f(x, y_i, z_i)\,\hat{\mathbf{i}} + \int_{y_i}^{y_f} dy f(x_f, y, z_i)\,\hat{\mathbf{j}} + \int_{z_i}^{z_f} dz f(x_f, y_f, z)\,\hat{\mathbf{k}}.\end{aligned}$$

The second expression holds for the specific path shown below with initial point $\mathbf{x}_i = (x_i, y_i, z_i)$ and final point $\mathbf{x}_f = (x_f, y_f, z_f)$. In general, the result depends on the path chosen.

If $f(\mathbf{x})$ is a constant, the integral is path-independent. For $f \equiv 1$ the integral becomes the distance vector between the endpoints of the path: $\mathbf{I} = \mathbf{x}_f - \mathbf{x}_i$.



▷ The line integral of a vector field $\mathbf{F}(\mathbf{x})$ yields a scalar I if it is constructed as follows:

$$\begin{aligned} I &= \int_C d\mathbf{x}\cdot\mathbf{F}(\mathbf{x}) \\ &= \int_{x_i}^{x_f} dx F_x(x,y_i,z_i) + \int_{y_i}^{y_f} dy F_y(x_f,y,z_i) + \int_{z_i}^{z_f} dz F_z(x_f,y_f,z). \end{aligned}$$

The second expression holds for the same specific path shown above. In general, the result again depends on the path chosen.

The integral is path-independent if the field is irrotational, $\nabla\times\mathbf{F}=0$, in which case it is the gradient of a scalar, $\mathbf{F}=\nabla f$. This implies that the differential *df* of the scalar field $f(\mathbf{x})$ is *exact*.

For this case, the integral along the path chosen becomes

$$\begin{aligned} I &= \int_{x_i}^{x_f} dx \frac{\partial f}{\partial x}\Big|_{y_i,z_i} + \int_{y_i}^{y_f} dy \frac{\partial f}{\partial y}\Big|_{x_f,z_i} + \int_{z_i}^{z_f} dz \frac{\partial f}{\partial z}\Big|_{x_f,y_f} \\ &= [f(x_f,y_i,z_i) - f(x_i,y_i,z_i)] + [f(x_f,y_f,z_i) - f(x_f,y_i,z_i)] \\ &\qquad + [f(x_f,y_f,z_f) - f(x_f,y_f,z_i)] \\ &= f(x_f,y_f,z_f) - f(x_i,y_i,z_i) = f(\mathbf{x}_i) - f(\mathbf{x}_f). \end{aligned}$$

The loop integral written in the form,

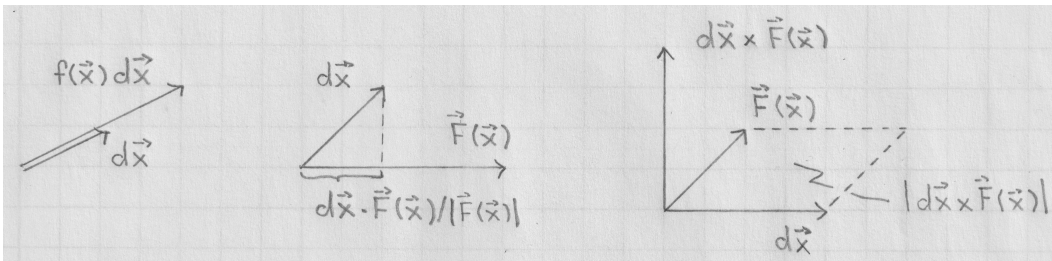
$$I = \oint_C d\mathbf{x}\cdot\mathbf{F}(\mathbf{x}),$$

is also known as *circulation*. The (default) positive sense of circulation is counterclockwise. Any loop integral for an irrotational field vanishes.

▷ The line integral of a vector function $\mathbf{F}(\mathbf{x})$ yields a vector $\mathbf{I}$ if it is constructed as cross product instead of a dot product:

$$\mathbf{I} = \int_C d\mathbf{x} \times \mathbf{F}(\mathbf{x}).$$

This figure illustrates the three kinds of line integrals in comparison.



Surface integrals:

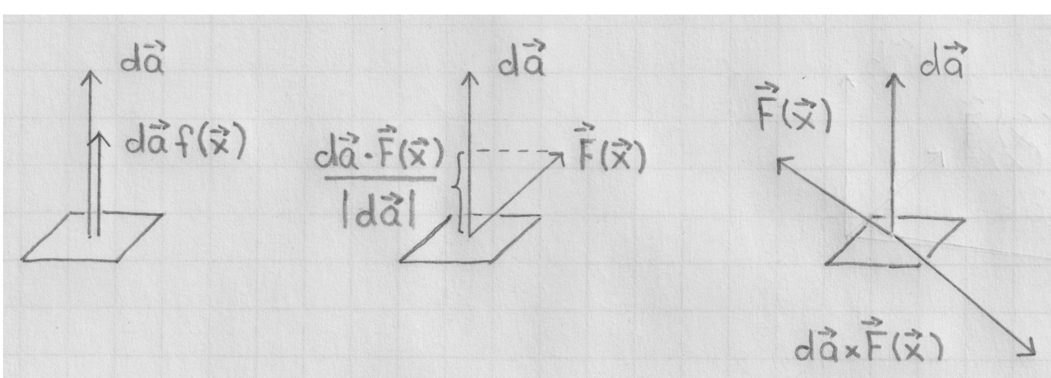
The centerpiece of surface integrals is the vector $d\mathbf{a}$ associated with elements of surface area. It is chosen sufficiently small to become essentially flat on the surface in question and directed perpendicular to it.

Surface integrals defined as follows then produce a vector, a scalar, and a vector again from left to right.

$$\mathbf{I} = \int_S d\mathbf{a} f(\mathbf{x}), \quad I = \int_S d\mathbf{a} \cdot \mathbf{F}(\mathbf{x}), \quad \mathbf{I} = \int_S d\mathbf{a} \times \mathbf{F}(\mathbf{x}).$$

In the case of an open surface S, one of two options must be chosen for the direction of the area vector. Closed surfaces have an inside and an outside. By convention, the vector of area element $d\mathbf{a}$ points toward the outside.

This figure illustrates the three kinds of surface integrals in comparison.



Note that each area element contributes a vector perpendicular to the surface in the first integral and a vector tangential to the surface in the third integral.

The second integral, which is a scalar, represents the flux associated with the field $\mathbf{F}(\mathbf{x})$ for the surface S.

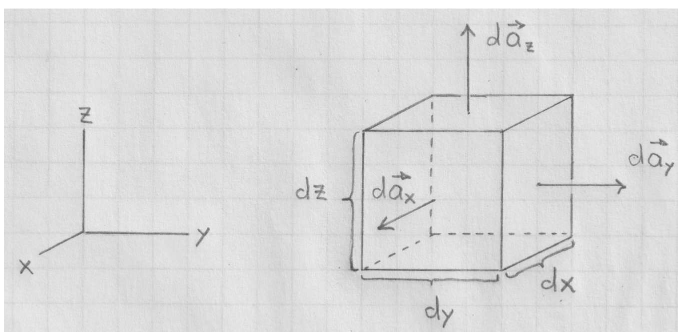
The third integral, which is a vector, makes a frequent appearance in magnetostatics, for example.

For completeness we note that the volume integral of a scalar (vector) function yields a scalar (vector) quantity.

Differential operators from integrals:

A cube of infinitesimal volume $V = dxdydz$ is positioned at $\mathbf{x} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$. Its six faces have area vectors (directed outward),

$$\pm d\mathbf{a}_x = \pm dydz\,\hat{\mathbf{i}}, \quad \pm d\mathbf{a}_y = \pm dxdz\,\hat{\mathbf{j}}, \quad \pm d\mathbf{a}_z = \pm dxdy\,\hat{\mathbf{k}}.$$



The integral of a scalar function $f(\mathbf{x})$ or a vector function $\mathbf{F}(\mathbf{x})$ over the surface of this cube combined with the limit $V \to 0$ can be used to reproduce the three elementary differential operators in Cartesian components:

– Gradient: $\nabla f = \lim_{V\to 0} \frac{1}{V} \oint_S d\mathbf{a}\, f,$

– Divergence: $\nabla \cdot \mathbf{F} = \lim_{V\to 0} \frac{1}{V} \oint_S d\mathbf{a} \cdot \mathbf{F}.$

– Curl: $\nabla \times \mathbf{F} = \lim_{V\to 0} \frac{1}{V} \oint_S d\mathbf{a} \times \mathbf{F}.$

▷ We have noted earlier that the first integral yields a vector. We can evaluate the surface integral by using differentials of f and the above area elements for pairs of opposite faces of the infinitesimal cube:

$$[f(x+dx,y,z) - f(x,y,z)]dydz$$
$$= \left[f(x,y,z) + \frac{\partial f}{\partial x}\,dx - f(x,y,z)\right] dydz = \frac{\partial f}{\partial x}\,dxdydz,$$

$$[f(x,y+dy,z) - f(x,y,z)]dzdx$$
$$= \left[f(x,y,z) + \frac{\partial f}{\partial y}\,dy - f(x,y,z)\right] dzdx = \frac{\partial f}{\partial y}\,dxdydz,$$

$$[f(x,y,z+dz) - f(x,y,z)]dxdy$$
$$= \left[f(x,y,z) + \frac{\partial f}{\partial z}\,dz - f(x,y,z)\right] dxdy = \frac{\partial f}{\partial z}\,dxdydz,$$

Add the three contributions vectorially and divide by the volume:

$$\frac{1}{V}\oint_S d\mathbf{a}\,f \rightsquigarrow \frac{1}{dxdydz}\left[dydz\frac{\partial f}{\partial x}dx\,\hat{\mathbf{i}} + dxdz\frac{\partial f}{\partial y}dy\,\hat{\mathbf{j}} + dxdy\frac{\partial f}{\partial z}dz\,\hat{\mathbf{k}}\right].$$
$$\Rightarrow\ \nabla f = \frac{\partial f}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\,\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\,\hat{\mathbf{k}}.$$

▷ For the evaluation of the second integral, which yields a scalar, we use differentials of the components of $\mathbf{F}$ and the above area vectors in combination with the elementary dot products,

$$\hat{\mathbf{i}}\cdot\mathbf{F} = F_x, \quad \hat{\mathbf{j}}\cdot\mathbf{F} = F_y, \quad \hat{\mathbf{k}}\cdot\mathbf{F} = F_z.$$

We can thus write for pairs of opposite faces of the infinitesimal cube:

$$\big[F_x(x+dx,y,z) - F_x(x,y,z)\big]dydz$$
$$= \left[F_x(x,y,z) + \frac{\partial F_x}{\partial x}\,dx - F_x(x,y,z)\right] dydz = \frac{\partial F_x}{\partial x}\,dxdydz$$

$$\big[F_y(x,y+dy,z) - F_y(x,y,z)\big]dzdx$$
$$= \left[F_y(x,y,z) + \frac{\partial F_y}{\partial y}\,dy - F_y(x,y,z)\right] dzdx = \frac{\partial F_y}{\partial y}\,dxdydz$$

$$\big[F_z(x,y,z+dz) - F_z(x,y,z)\big]dxdy$$
$$= \left[F_z(x,y,z) + \frac{\partial F_z}{\partial z}\,dz - F_z(x,y,z)\right] dxdy = \frac{\partial F_z}{\partial z}\,dxdydz.$$

Add the three (scalar) contributions and divide by the volume:

$$\Rightarrow \; \frac{1}{V}\oint_S d\mathbf{a}\cdot\mathbf{F} \rightsquigarrow \frac{1}{dxdydz}\left[dydz\frac{\partial F_x}{\partial x}dx + dxdz\frac{\partial F_y}{\partial y}dy + dxdy\frac{\partial F_z}{\partial z}dz\right]$$

$$\Rightarrow \; \nabla\cdot\mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}.$$

▷ The third integral, which yields a vector, is evaluated along the same lines. Here we use the elementary cross products,

$$\hat{\mathbf{i}}\times\mathbf{F} = F_y\,\hat{\mathbf{k}} - F_z\,\hat{\mathbf{j}}, \quad \hat{\mathbf{j}}\times\mathbf{F} = F_z\,\hat{\mathbf{i}} - F_x\,\hat{\mathbf{k}}, \quad \hat{\mathbf{k}}\times\mathbf{F} = F_x\,\hat{\mathbf{j}} - F_y\,\hat{\mathbf{i}},$$

the same expansions for the components of **F** and assemble them accordingly:

$$\frac{1}{V}\oint_S d\mathbf{a}\times\mathbf{F} \rightsquigarrow \frac{1}{dxdydz}\Bigg[dxdy\left(\frac{\partial F_y}{\partial x}\,\hat{\mathbf{k}} - \frac{\partial F_z}{\partial x}\,\hat{\mathbf{j}}\right)dx$$
$$+\, dxdz\left(\frac{\partial F_z}{\partial y}\,\hat{\mathbf{i}} - \frac{\partial F_x}{\partial y}\,\hat{\mathbf{k}}\right)dy + dydz\left(\frac{\partial F_x}{\partial z}\,\hat{\mathbf{j}} - \frac{\partial F_y}{\partial z}\,\hat{\mathbf{i}}\right)dz\Bigg].$$

$$\Rightarrow \; \nabla\times\mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{\mathbf{k}}.$$

Integral theorems:

The best known and most widely used integral theorems are *Stokes' theorem* and *Gauss's theorem.*

We begin with the *gradient theorem* for the sake of completing a pattern and end with *Green's theorem*, a widely used special case of Stokes' theorem applied to vectors in a plane.

▷ The gradient theorem is an integral version of the identity $\nabla \times (\nabla f) = 0$, stated in two versions as follows:

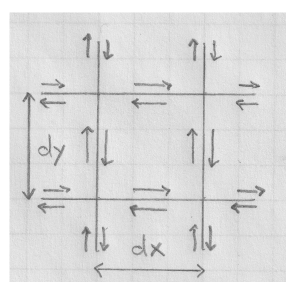
$$\int_{\mathbf{x}_0}^{\mathbf{x}_1} d\mathbf{x} \cdot \nabla f = f(\mathbf{x}_1) - f(\mathbf{x}_0) \quad \text{(independent of the path chosen)},$$

$$\oint d\mathbf{x} \cdot \nabla f = 0 \quad \text{(for any closed loop)}.$$

▷ Stokes' theorem relates the circulation of vector field $\mathbf{F}$ along the loop (closed path) C to the integral of the scalar flux quantity constructed from the vector $\nabla \times \mathbf{F}$ and the element $d\mathbf{a}$ of an open surface S with perimeter C:

$$\oint_C d\mathbf{l} \cdot \mathbf{F} = \int_S d\mathbf{a} \cdot (\nabla \times \mathbf{F}).$$

The surface integral is unique, even though the surface S for given perimeter C is not. The proof for a flat surface (in the xy-plane) can be constructed by dividing the surface into infinitesimal squares. The loop integral around each square of area $da = dxdy$ evaluated by use of the differential of $\mathbf{F}$ becomes



$$\oint_{\square} d\mathbf{l} \cdot \mathbf{F} = \int_y^{y+dy} dy[F_y(x+dx,y) - F(x,y)]$$
$$+ \int_x^{x+dx} dx[F_x(x,dy) - F(x,y+dy)]$$
$$\rightsquigarrow dy\,\frac{\partial F_y}{\partial x}\,dx - dx\,\frac{\partial F_x}{\partial y}\,dy = da\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) = d\mathbf{a} \cdot (\nabla \times \mathbf{F})_z$$

All line integrals over interior sides of squares cancel.

▷ Gauss's theorem (also named divergence theorem) relates the flux of the vector field $\mathbf{F}$ through the closed surface S to the integral of the scalar quantity $\nabla \cdot \mathbf{F}$ over the (unique) interior volume V:

$$\oint_S d\mathbf{a} \cdot \mathbf{F} = \int_V d^3x \, \nabla \cdot \mathbf{F}.$$

The proof can be constructed by dividing the volume V into infinitesimal cubes and apply the second integral of the previous section. The surface integrals over all interior walls cancel.

▷ Green's theorem is a special case of Stokes' theorem confined to two-component vectors in a plane:

$$\oint_C [M(x,y)dx + N(x,y)dy] = \int_R dxdy \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right],$$

where C is the simple perimeter loop of the compact region R.

In a sense, the first three theorems show a pattern of progression in spatial dimension of the same theme.

– The region of integration progresses from a line segment to a compact region on a surface to a compact volume in space.

– The boundary progresses from two endpoints to a surrounding loop to an enclosing surface.

– The gradient theorem relates the integral of the gradient of scalar function across a compact 1D region to the function evaluated at the endpoints.

– Stokes' theorem relates the integral of the curl of a vector function across a compact 2D region to the vector function evaluated on points of the surrounding loop.

– Gauss's theorem relates the integral of the divergence of a vector function across a compact 3D region to a the vector function evaluated on points of the enclosing surface.

Green's identities:

Consider a region of space V bounded by a surface S. Two vector fields, $\phi\nabla\psi$ and $\psi\nabla\phi$, in that region are derived from a pair of scalar fields ϕ, ψ.

Next we apply a mathematical identity introduced earlier for the divergence of the vector fields thus constructed:

$$\nabla \cdot (\phi\nabla\psi) = \phi\nabla^2\psi + \nabla\phi \cdot \nabla\psi, \quad \nabla \cdot (\psi\nabla\phi) = \psi\nabla^2\phi + \nabla\psi \cdot \nabla\phi.$$

Then we integrate the two relationss over the region and use Gauss's theorem to arrive at *Green's first identity* in two versions:

$$\int_V d^3x \big[\phi\nabla^2\psi + \nabla\phi \cdot \nabla\psi\big] = \oint_S da\, \phi\, \frac{\partial\psi}{\partial n},$$
$$\int_V d^3x \big[\psi\nabla^2\phi + \nabla\psi \cdot \nabla\phi\big] = \oint_S da\, \psi\, \frac{\partial\phi}{\partial n},$$

in the process, we have set the stage for the surface integration as follows:

$$\phi\nabla\psi \cdot \hat{\mathbf{n}} = \phi\, \frac{\partial\psi}{\partial n}, \quad \psi\nabla\phi \cdot \hat{\mathbf{n}} = \psi\, \frac{\partial\phi}{\partial n}.$$

Green's second identity results from the difference of the two versions of the first identity, which eliminates the gradient product terms:

$$\int_V d^3x \big[\phi\nabla^2\psi - \psi\nabla^2\phi\big] = \oint_S da \left[\phi\, \frac{\partial\psi}{\partial n} - \psi\, \frac{\partial\phi}{\partial n}.\right].$$

Integration by parts generalized:

From identities introduced above we can infer relations between different integrals involving products of scalar and vector functions. In each case, one of the three integrals is amenable to one of the integral theorems.

$$\int_V d^3x\, f(\nabla \cdot \mathbf{G}) + \int_V d^3x\, \mathbf{G} \cdot (\nabla f) \\ = \int_V d^3x\, \nabla \cdot (f\mathbf{G}) \overset{\text{Gauss}}{=} \oint_S d\mathbf{a} \cdot (f\mathbf{G}).$$

$$\int_S d\mathbf{a} \cdot f(\nabla \times \mathbf{G}) + \int_S d\mathbf{a} \cdot [(\nabla f) \times \mathbf{G}] \\ = \int_S d\mathbf{a} \cdot [\nabla \times (f\mathbf{G})] \overset{\text{Stokes}}{=} \oint_C d\mathbf{l} \cdot (f\mathbf{G}).$$

$$\int_V d^3x (\nabla \times \mathbf{F}) \cdot \mathbf{G} - \int_V d^3x\, \mathbf{F} \cdot (\nabla \times \mathbf{G}) \\ = \int_V d^3x\, \nabla \cdot (\mathbf{F} \times \mathbf{G}) \overset{\text{Gauss}}{=} \oint_S d\mathbf{a} \cdot (\mathbf{F} \times \mathbf{G}).$$

Helmholtz theorem:

Consider a vector field $\mathbf{H}(\mathbf{x})$ of which the divergence and curl are given:

$$\nabla \cdot \mathbf{H}(\mathbf{x}) = d(\mathbf{x}), \quad \nabla \times \mathbf{H}(\mathbf{x}) = \mathbf{c}(\mathbf{x}).$$

The Helmholtz theorem states that these specifications uniquely determine the function $\mathbf{H}(\mathbf{x})$ under mild conditions.

Unique decomposition: $\mathbf{H}(\mathbf{x}) = -\nabla\psi(\mathbf{x}) + \nabla \times \mathbf{A}(\mathbf{x})$, where

$$\psi(\mathbf{x}) = \frac{1}{4\pi}\int d^3x' \frac{d(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}, \quad \mathbf{A}(\mathbf{x}) = \frac{1}{4\pi}\int d^3x' \frac{\mathbf{c}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}.$$

Here we show that the decomposition reproduces what is given.

▷ Show that $\nabla \cdot \mathbf{H}(\mathbf{x}) = d(\mathbf{x})$.

– Use identity: $\nabla \cdot (\nabla \times \mathbf{A}) = 0$.

– Consequence: $-\nabla^2\psi = d(\mathbf{x})$.

– Use identity: $\nabla^2 \left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = -4\pi\delta(\mathbf{x}-\mathbf{x}')$.

– Application to integral expression:

$$-\nabla^2\psi = -\frac{1}{4\pi}\int d^3x'\, \nabla^2 \frac{d(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} = \int d^3x'\, d(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}') = d(\mathbf{x}).$$

▷ Show that $\nabla \times \mathbf{H}(\mathbf{x}) = \mathbf{c}(\mathbf{x})$.

– Use identity: $\nabla \times (\nabla\psi) = 0$.

– Consequence: $\nabla \times (\nabla \times \mathbf{A}) = \mathbf{c}(\mathbf{x})$.

– Use identity: $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$.

– Apply Laplacian to integral expression:

$$-\nabla^2\mathbf{A} = -\frac{1}{4\pi}\int d^3x'\, \nabla^2 \frac{\mathbf{c}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} = \int d^3x'\, \mathbf{c}(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}') = \mathbf{c}(\mathbf{x}).$$

– Show that $\nabla \cdot \mathbf{A} = 0$:

$$4\pi\, \nabla \cdot \mathbf{A} = \int d^3x'\, \nabla \frac{1}{|\mathbf{x}-\mathbf{x}'|} \cdot \mathbf{c}(\mathbf{x}') = -\int d^3x' \left[\nabla' \frac{1}{|\mathbf{x}-\mathbf{x}'|}\right] \cdot \mathbf{c}(\mathbf{x}')$$
$$= -\int d^3x' \left(\nabla' \cdot \left[\frac{\mathbf{c}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right] - \frac{\nabla' \cdot \mathbf{c}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right).$$

The second integrand vanishes by construction: $\nabla \cdot \mathbf{c} \equiv 0$. The integral of the first term vanishes on account of Gauss's theorem (surface integral at infinity).