Vector Analysis I [gmd1-A]

The vectors considered here for the most part are quantities with magnitude and direction in 3-dimensional Euclidean space.

Vector algebra is about addition and multiplications of 3-component vectors. *Vector analysis* also includes differential and integral operations with vectors.

Vectors in the 3-dimensional vector space $\mathbb{R}^3$ have three components. Generalizations to the vector space $\mathbb{R}^n$ and other spaces are discussed elsewhere.

A *matrix* with a single row is known as *row vector* and a matrix with a single column as a *column vector*. Matrix operations are discussed elsewhere.

A *tensor* of rank one is also known as a vector. The tensor attributes of vectors are among the topics of a different module.

Expressions with vectors have a different look in different coordinate systems (e.g. rectangular, cylindrical and spherical coordinates). The choice of coordinate system is informed by symmetry.

The focus here is on coordinate-independent expressions (geometric representations). Only Cartesian components are used. Coordinate systems and coordinate transformations are discussed in detail elsewhere.

Vector addition:

Cartesian unit vectors: $\hat{\mathbf{i}},\ \hat{\mathbf{j}},\ \hat{\mathbf{k}}$.

Vector in Cartesian components: $\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}} = (A_x, A_y, A_z)$.

Norm (magnitude): $A = |\mathbf{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$ (scalar quantity).

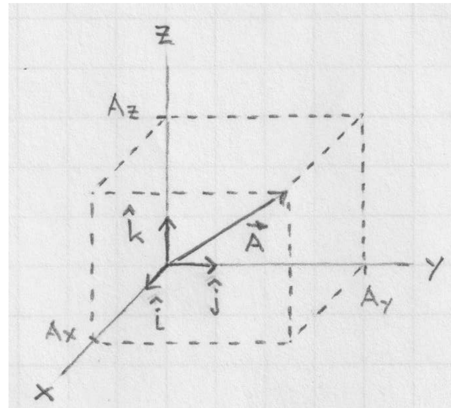
Null vector, $\mathbf{0} = (0, 0, 0)$, has zero magnitude and no direction.

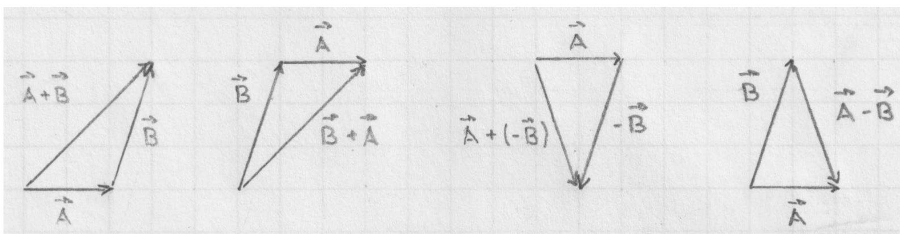
Sum: $\mathbf{A} + \mathbf{B} = (A_x + B_x)\hat{\mathbf{i}} + (A_y + B_y)\hat{\mathbf{j}} + (A_z + B_z)\hat{\mathbf{k}}$.

Vector addition is commutative and associative. The vectors $\mathbf{A}$ and $-\mathbf{A}$ have the same magnitude and opposite direction. Subtraction of $\mathbf{B}$ means addition of $-\mathbf{B}$. A vector multiplied by a scalar a (real number) changes its magnitude if $a \neq \pm 1$ and switches its direction if $a < 0$.

Axioms of a vector space (including $\mathbb{R}^3$):

- ▷ $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.
- ▷ $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$.
- ▷ $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$.
- ▷ $\mathbf{A} + \mathbf{0} = \mathbf{A}$.
- ▷ $a(\mathbf{A} + \mathbf{B}) = a\mathbf{A} + a\mathbf{B}$.
- ▷ $(a + b)\mathbf{A} = a\mathbf{A} + b\mathbf{A}$.
- ▷ $(ab)\mathbf{A} = a(b\mathbf{A})$.
- ▷ $1\mathbf{A} = \mathbf{A}$.





A systematic description of vector spaces, Banach spaces, Hilbert spaces, and more is a topic of *functional analysis* [gmd12].

Dot product of vectors:

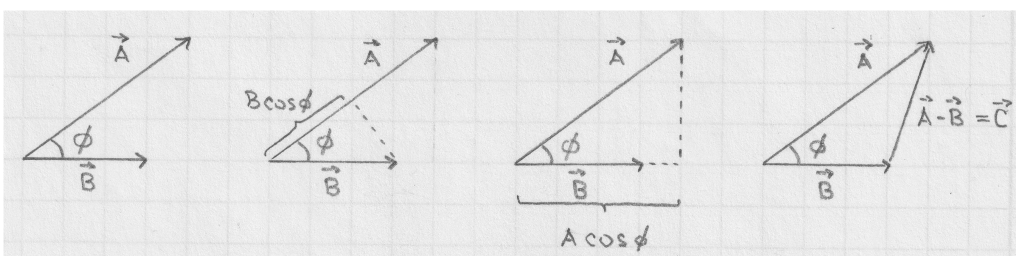
The dot product yields a scalar. It is also named *scalar product*. The dot product is commutative. Perpendicular vectors yield zero. Parallel vectors yield the product of magnitudes.

The magnitude of a vector, the angle between two vectors, and the law of cosines can be inferred from dot products.

- ▷ $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$.
- ▷ $\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$.
- ▷ $\mathbf{A} \cdot (c\mathbf{B}) = c(\mathbf{A} \cdot \mathbf{B})$.
- ▷ $\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1, \quad \hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{i}} = 0$.
- ▷ $\mathbf{A} \cdot \mathbf{B} = (A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}) \cdot (B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}) = A_xB_x + A_yB_y + A_zB_z$.
- ▷ $A = |\mathbf{A}| = \sqrt{\mathbf{A} \cdot \mathbf{A}} = \sqrt{A_x^2 + A_y^2 + A_z^2}$ (magnitude).
- ▷ $\mathbf{A} \cdot \mathbf{B} = AB\cos\phi$.
- ▷ $(\mathbf{A} - \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B}) = \mathbf{A} \cdot \mathbf{A} + \mathbf{B} \cdot \mathbf{B} - 2\mathbf{A} \cdot \mathbf{B} = A^2 + B^2 - 2AB\cos\phi$.

The last relation is illustrated by the triangle on the right.

Law of cosines: $C^2 = A^2 + B^2 - 2AB\cos\phi$.

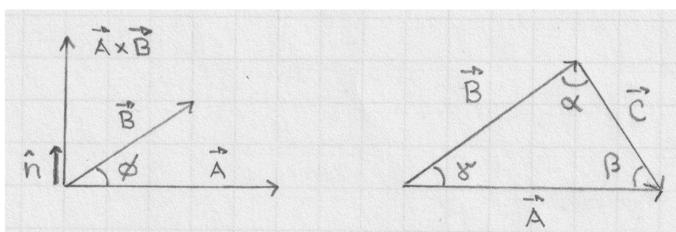


Cross product of vectors:

The cross product yields a vector. It is also named *vector product*. Exchanging factors switches the direction of the product. The vector $\mathbf{A} \times \mathbf{B}$ is perpendicular to the plane of $\mathbf{A}$ and $\mathbf{B}$. If $\mathbf{A}$ and $\mathbf{B}$ are parallel (same or opposite direction) then $\mathbf{A} \times \mathbf{B} = \mathbf{0}$.

- $\triangleright$ $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$.
- $\triangleright$ $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$.
- $\triangleright$ $\mathbf{A} \times (c\mathbf{B}) = (c\mathbf{A}) \times \mathbf{B} = c(\mathbf{A} \times \mathbf{B})$.
- $\triangleright$ $\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{k}} = 0, \quad \hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}, \quad \hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}, \quad \hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$.
- $\triangleright$ $\mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y)\hat{\mathbf{i}} + (A_z B_x - A_x B_z)\hat{\mathbf{j}} + (A_x B_y - A_y B_x)\hat{\mathbf{k}}$.
- $\triangleright$ $\mathbf{A} \times \mathbf{B} = AB \sin\phi\, \hat{\mathbf{n}} \quad (\hat{\mathbf{n}} \perp \mathbf{A},\ \hat{\mathbf{n}} \perp \mathbf{B}$, right-hand rule)
- $\triangleright$ $\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$.

The area of the parallelogram with sides $\mathbf{A}$ and $\mathbf{B}$ is equal to $|\mathbf{A} \times \mathbf{B}|$.



The law of sines is derived from three equivalent expressions of the area of the triangle on the right:

$$\frac{1}{2}|\mathbf{A} \times \mathbf{B}| = \frac{1}{2}|\mathbf{B} \times \mathbf{C}| = \frac{1}{2}|\mathbf{C} \times \mathbf{A}| \quad \Rightarrow \ AB\sin\gamma = BC\sin\alpha = CA\sin\beta.$$

$$\Rightarrow \ \frac{\sin\alpha}{A} = \frac{\sin\beta}{B} = \frac{\sin\gamma}{C} \quad \text{(law of sines)}.$$

Triple products of vectors:

There three meaningful products of three vectors **A**, **B**, **C**.

▷ $(\mathbf{A}\cdot\mathbf{B})\mathbf{C} \neq \mathbf{A}(\mathbf{B}\cdot\mathbf{C})$.

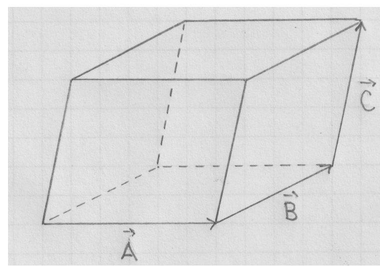
This product is the vector outside the parentheses multiplied by the scalar product of the other two vectors.

▷ $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$ (triple scalar product).

This product yields a scalar from three vectors. It is invariant under cyclic permutation of the factors.

Geometrically, if the vectors **A**, **B**, **C**, are the esges from one corner of a parallelepiped, then the triple scalar product is ± its volume:

$$\pm V = \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}.$$



Three mutually orthogonal vectors **A**, **B**, **C** form a right-handed triad if $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) > 0$. The unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$ satisfy this condition.

▷ $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$ (triple vector product).

This product yields a vector which must be perpendicular to $\mathbf{B}\times\mathbf{C}$, implying that it is in the plane of **B** and **C**.

The parenthesis in the first and second trile product are optional. There is only one way to sequence the operations. The parentheses in the triple vector producct are not optional: $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) \neq (\mathbf{A}\times\mathbf{B})\times\mathbf{C}$, in general.

Useful identities involving multiple vector or scalar products:

$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C}).$$

$$\begin{aligned}(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) &= (\mathbf{A}\cdot\mathbf{B}\times\mathbf{D})\mathbf{C} - (\mathbf{A}\cdot\mathbf{B}\times\mathbf{C})\mathbf{D} \\ &= (\mathbf{A}\cdot\mathbf{C}\times\mathbf{D})\mathbf{B} - (\mathbf{B}\cdot\mathbf{C}\times\mathbf{D})\mathbf{A}.\end{aligned}$$

Reciprocal vectors:

Consider three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ which are not coplanar and form a right-handed triad.

A set of *reciprocal vectors* $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ must, by definition, satisfy the mutual orthonormality conditions:

$$\mathbf{a}'\cdot\mathbf{a} = \mathbf{b}'\cdot\mathbf{b} = \mathbf{c}'\cdot\mathbf{c} = 1 \quad \mathbf{a}'\cdot\mathbf{b} = \mathbf{a}'\cdot\mathbf{c} = \mathbf{b}'\cdot\mathbf{a} = \mathbf{b}'\cdot\mathbf{c} = \mathbf{c}'\cdot\mathbf{a} = \mathbf{c}'\cdot\mathbf{b} = 0.$$

The reciprocal vectors can be constructed in the form,

$$\mathbf{a}' = \frac{1}{V}\,\mathbf{b}\times\mathbf{c}, \quad \mathbf{b}' = \frac{1}{V}\,\mathbf{c}\times\mathbf{a}. \quad \mathbf{c}' = \frac{1}{V}\,\mathbf{a}\times\mathbf{b},$$

where the volume of the parallelepiped spanned by $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is

$$V = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}).$$

We can thus write:

$$\mathbf{a}' = \frac{\mathbf{b}\times\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}}, \quad \mathbf{b}' = \frac{\mathbf{c}\times\mathbf{a}}{\mathbf{b}\cdot\mathbf{c}\times\mathbf{a}}, \quad \mathbf{c}' = \frac{\mathbf{a}\times\mathbf{b}}{\mathbf{c}\cdot\mathbf{a}\times\mathbf{b}}.$$

The orthogonality relations are readily confirmed.

Reciprocity cuts both ways, meaning that the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ must be recovered if the above expressions for $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ are substituted into the right-hand side of the relations,

$$\mathbf{a} = \frac{1}{v}\,\mathbf{b}'\times\mathbf{c}', \quad \mathbf{b} = \frac{1}{v}\,\mathbf{c}'\times\mathbf{a}'. \quad \mathbf{c} = \frac{1}{v}\,\mathbf{a}'\times\mathbf{b}',$$

where

$$v = \mathbf{a}'\cdot(\mathbf{b}'\times\mathbf{c}') = \mathbf{b}'\cdot(\mathbf{c}'\times\mathbf{a}') = \mathbf{c}'\cdot(\mathbf{a}'\times\mathbf{b}').$$

The volumes V and v are also reciprocal as it turns out [gex28].

In the limit where the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ become orthonormal unit vectors, the reciprocal vectors $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ are identical to the original ones and $V = v = 1$.

Reciprocal vectors play an important role in the theory of crystal lattices. When vectors are expanded in a non-orthornormal basis, reciprocal vectors come into play [gex29].

Vector functions:

If a vector $\mathbf{A}$ depends on a continuous parameter u we have a vector-valued function $\mathbf{A}(u)$. The parametrized vector,

$$\mathbf{x}(u) = x(u)\,\hat{\mathbf{i}} + y(u)\,\hat{\mathbf{j}} + z(u)\,\hat{\mathbf{k}},$$

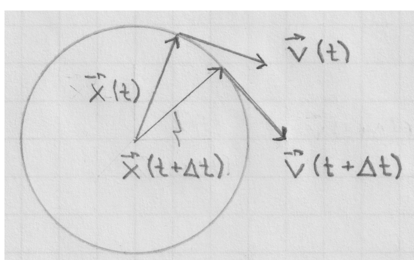
describes a *space curve.*

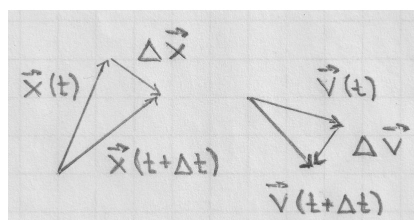
Consider the case of position as a (differentiable) function of time:

- Trajectory: $\mathbf{x}(t) = x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}} + z(t)\,\hat{\mathbf{k}}$.
- Velocity: $\mathbf{v}(t) \doteq \lim_{\Delta t \to 0} \dfrac{\Delta \mathbf{x}}{\Delta t} = \dfrac{d\mathbf{x}}{dt}$.
- Acceleration: $\mathbf{a}(t) \doteq \lim_{\Delta t \to 0} \dfrac{\Delta \mathbf{v}}{\Delta t} = \dfrac{d\mathbf{v}}{dt}$.

This vector function $\mathbf{x}(t)$ traces a trajectory in space, represented geometrically by an evolving space curve.

The velocity vector $\mathbf{v}(t)$ is tangential to the trajectory in position space. It traces a trajectory in velocity space to which the acceleration vector $\mathbf{a}(t)$ is tangential.





Case of circular motion at constant speed:

- *Position space*: $\mathbf{x}(t)$ traces a circle pointing radially out. $\mathbf{v}(t)$ is tangential to that circle in foward direction. $\mathbf{a}(t)$ is directed radially in.
- *Velocity space*: $\mathbf{v}(t)$ traces a circle pointing radially out. $\mathbf{a}(t)$ is oriented tangentially to that circle.

The analysis of space curves is a topic of *differential geometry* [gmd13].

Scalar and vector fields:

Functions of spatial coordinates are known as *fields*.

– Scalar field: $f(\mathbf{x}) = f(x,y,z), \quad \mathbf{x} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$.
– Vector field: $\mathbf{F}(\mathbf{x}) = F_x(x,y,z)\,\hat{\mathbf{i}} + F_y(x,y,z)\,\hat{\mathbf{j}} + F_z(x,y,z)\,\hat{\mathbf{k}}$.

Fields may depend on time. A time-dependent scalar field changes its value with time at all points in space. A time-dependent vector field changes magnitude and direction at each location as time evolves.

Differential operators:

Differential operator:[1] $\nabla = \frac{\partial}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial}{\partial y}\,\hat{\mathbf{j}} + \frac{\partial}{\partial z}\,\hat{\mathbf{k}}$.

Gradient: $\nabla f = \frac{\partial f}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\,\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\,\hat{\mathbf{k}}$.

Divergence: $\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$.

Curl: $\nabla \times \mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{\mathbf{k}}$.

Laplacian: $\nabla^2 f = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$.

These expressions are formulated for Cartesian coordinate systems. Different coordinate systems will be considered elsewhere [gmd2]. The linearity of these differential operators enables the superposition principle.

The gradient of a scalar field is a vector field. The divergence of a vector field is a scalar field. The curl of a vector field is another vector field. The Laplacian a scalar field is another scalar field.

The gradient ∇f encodes the direction and slope of steepest ascent. The directional derivative, $\nabla f \cdot \mathbf{n}$, encodes the rate of ascent in direction $\mathbf{n}$.

Local extrema of f (maxima, minima, saddle points) have $\nabla f = \mathbf{0}$. Local extrema subject to auxiliary conditions can be found via the vanishing gradient of an extended field expression with a Lagrange multiplier [gex31].

[1]The symbol of the operator ∇ is named ‘nabla’. When used as a gradient, divergence, or curl, we say ‘’del f’, ‘del dot $\mathbf{F}$’, and ‘del cross $\mathbf{F}$’, respectively.

Identities involving differential operators:

Differential operators: gradient, divergence, curl, Laplacian.

Scalar functions: $f(\mathbf{x})$, $g(\mathbf{x})$.

Vector functions: $\mathbf{F}(\mathbf{x})$, $\mathbf{G}(\mathbf{x})$.

Derivatives of products:[2]

- ▷ $\nabla(fg) = f(\nabla g) + g(\nabla f)$
- ▷ $\nabla(\mathbf{F} \cdot \mathbf{G}) = \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F}) + (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F}$
- ▷ $\nabla \cdot (g\mathbf{F}) = g(\nabla \cdot \mathbf{F}) + (\nabla g) \cdot \mathbf{F}$
- ▷ $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\nabla \times \mathbf{G})$
- ▷ $\nabla \times (g\mathbf{F}) = g(\nabla \times \mathbf{F}) + (\nabla g) \times \mathbf{F}$
- ▷ $\nabla \times (\mathbf{F} \times \mathbf{G}) = (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G} + \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F})$

Derivatives of quotients:

- ▷ $\nabla \left(\frac{f}{g}\right) = \frac{g(\nabla f) - f(\nabla g)}{g^2}$
- ▷ $\nabla \cdot \left(\frac{\mathbf{F}}{g}\right) = \frac{g(\nabla \cdot \mathbf{F}) - \mathbf{F} \cdot (\nabla g)}{g^2}$
- ▷ $\nabla \times \left(\frac{\mathbf{F}}{g}\right) = \frac{g(\nabla \times \mathbf{F}) - \mathbf{F} \times (\nabla g)}{g^2}$

Products of derivatives:[3]

- ▷ $\nabla \cdot \nabla f = \nabla^2 f$
- ▷ $\nabla \times (\nabla f) = 0$
- ▷ $\nabla \cdot (\nabla \times \mathbf{F}) = 0$
- ▷ $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$

Vectors ∇f are named *irrotational* and have zero curl.

Vectors $\nabla \times \mathbf{F}$ are named *solenoidal* and have zero divergence.

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[2] In the expression $(\mathbf{F} \cdot \nabla)\mathbf{G}$, the scalar operator $\mathbf{F} \cdot \nabla = F_x(\partial/\partial x) + F_y(\partial/\partial y) + F_z(\partial/\partial z)$ acts on the vector $\mathbf{G}$ and thus yields a vector.

[3] The Laplacian operating on a vector, $\nabla^2 \mathbf{F}$, has a straightforward meaning for Cartesian components: it acts on each component of $\mathbf{F}$ to produce the components of $\nabla^2 \mathbf{F}$. For curvilinear coordinates, the last identity can be used as the definition of $\nabla^2 \mathbf{F}$.

Differentials of scalars:

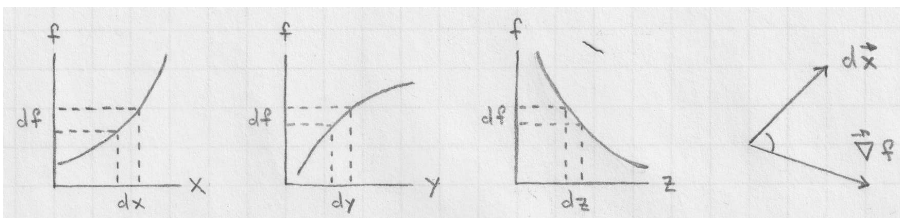
The differential of a scalar function $f(\mathbf{x})$ is constructed as a dot product from two vectors: its gradient and an infinitesimal displacement vector:

$$df = d\mathbf{x} \cdot \nabla f, \quad d\mathbf{x} = dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}} + dz\,\hat{\mathbf{k}}.$$

Here the scalar f is acted on by the scalar operator,

$$d\mathbf{x} \cdot \nabla = dx\,\frac{\partial}{\partial x} + dy\,\frac{\partial}{\partial y} + dz\,\frac{\partial}{\partial x},$$

which becomes the scalar df. If $d\mathbf{x} \cdot \nabla f = 0$, $d\mathbf{x}$ is along a line of constant $f(\mathbf{x})$. The vector ∇f points in the direction of steepest ascent of $f(\mathbf{x})$.



The differential df thus derived is named *exact* differential. A more general differential has the form,

$$dg = g_x(\mathbf{x})dx + g_y(\mathbf{x})dy + g_z(\mathbf{x})dz,$$

with arbitrary functions g_x, g_y, g_z in the role of coefficients. Such a differential is, in general, *inexact.*

A set of coefficients g_x, g_y, g_z specify an exact differential if they are the components of an irrotational vector, $\nabla \times \mathbf{g} = 0$, which implies the conditions,

$$\frac{\partial g_y}{\partial x} = \frac{\partial g_x}{\partial y}, \quad \frac{\partial g_z}{\partial y} = \frac{\partial g_y}{\partial z}, \quad \frac{\partial g_x}{\partial z} = \frac{\partial g_z}{\partial x}.$$

Any irrotational vector can be expressed as the gradient of a scalar, $\mathbf{g} = \nabla f$, implying the conditions,

$$g_x = \frac{\partial f}{\partial x}, \quad g_y = \frac{\partial f}{\partial y}, \quad g_z = \frac{\partial f}{\partial z}.$$

The difference between exact and inexact differentials matters for their integration between points or around a loop in space (a later topic).

Differentials of vectors:

The differential of a vector function $\mathbf{F}(\mathbf{x})$ is constructed as follows:

$$d\mathbf{F} = \mathbf{F}(\mathbf{x} + d\mathbf{x}) - \mathbf{F}(\mathbf{x}) = (d\mathbf{x} \cdot \nabla)\mathbf{F}.$$

Here the vector $\mathbf{F}$ is acted on by the scalar operator,

$$d\mathbf{x} \cdot \nabla = dx\,\frac{\partial}{\partial x} + dy\,\frac{\partial}{\partial y} + dz\,\frac{\partial}{\partial z},$$

producing the vector,

$$\begin{aligned} d\mathbf{F} = \left(dx\,\frac{\partial F_x}{\partial x} + dy\,\frac{\partial F_x}{\partial y} + dz\,\frac{\partial F_x}{\partial z}\right)\hat{\mathbf{i}} + \left(dx\,\frac{\partial F_y}{\partial x} + dy\,\frac{\partial F_y}{\partial y} + dz\,\frac{\partial F_y}{\partial z}\right)\hat{\mathbf{j}} \\ + \left(dx\,\frac{\partial F_z}{\partial x} + dy\,\frac{\partial F_z}{\partial y} + dz\,\frac{\partial F_z}{\partial z}\right)\hat{\mathbf{k}} \end{aligned}$$

The differential of position $\mathbf{x}$ simplifies into $d\mathbf{x} = dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}} + dz\,\hat{\mathbf{k}}$.

The differential of an irrotational vector $\mathbf{F}$, i.e. a vector which satisfies $\nabla \times \mathbf{F} = 0$, can be simplified as follows:

$$\begin{aligned} d\mathbf{F} &= \left(dx\,\frac{\partial F_x}{\partial x} + dy\,\frac{\partial F_y}{\partial x} + dz\,\frac{\partial F_z}{\partial x}\right)\hat{\mathbf{i}} + \left(dx\,\frac{\partial F_x}{\partial y} + dy\,\frac{\partial F_y}{\partial y} + dz\,\frac{\partial F_z}{\partial y}\right)\hat{\mathbf{j}} \\ &\quad + \left(dx\,\frac{\partial F_x}{\partial z} + dy\,\frac{\partial F_y}{\partial z} + dz\,\frac{\partial F_z}{\partial z}\right)\hat{\mathbf{k}} \\ &= \left(d\mathbf{x} \cdot \frac{\partial}{\partial x}\,\mathbf{F}\right)\hat{\mathbf{i}} + \left(d\mathbf{x} \cdot \frac{\partial}{\partial y}\,\mathbf{F}\right)\hat{\mathbf{j}} + \left(d\mathbf{x} \cdot \frac{\partial}{\partial z}\,\mathbf{F}\right)\hat{\mathbf{k}} \\ &= dx\nabla F_x + dy\nabla F_y + dz\nabla F_z. \end{aligned}$$