Linear ODEs [gam8]

General structure of ODE and solution:

Consider an n^{th}-order linear ODE for the function $y(x)$ in the form,

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = R(x).$$

Homogeneous ODEs have $R(x) \equiv 0$. Under mild assumptions, a unique solution exists which satisfies the following conditions or equivalent conditions:

$$y(x_0) = y_0, \quad y'(x_0) = y_0', \quad \ldots, \quad y^{(n-1)}(x_0) = y_0^{(n-1)}.$$

The following operator notation is sometimes being used, where D^i are linear differential operators:

$$D^i \doteq \frac{d^n}{dx^i}, \quad \Phi(D) \doteq \sum_{i=0}^{n} a_i(x)D^{n-i}. \quad \Phi(D)y = R(x).$$

Fundamental theorem of linear ODEs: The general solution of the inhomogeneous ODE can be stated as the general solution of the homogenous ODE (named complementary solution) plus any particular solution of the inhomogeneous ODE:

$$y(x) = Y_{\text{c}}(x) + Y_{\text{p}}(x).$$

The difference between any solutions $Y_{\text{p}}(x)$ is a solution included in $Y_{\text{c}}(x)$ – a consequence of the superposition principle.

The complementary solution includes n integration constants and can be expressed as a linear combination of the form,

$$Y_{\text{c}}(x) = \sum_{i=1}^{n} c_i y_i(x).$$

The linear independence of the functions $y_i(x)$ which make up the complementary solution is guaranteed by a nonvanishing *Wronskian determinant*:

$$W(y_1, \ldots, y_n) \doteq \begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{vmatrix}$$

The solution of a linear ODE entails significant simplifications for cases with constant coefficients a_i.

Homogeneous ODE with constant coefficients:

$$[a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n] y = 0.$$

Exponential ansatz for complementary solution: $y(x) = e^{mx}$.

Substitution produces characteristic polynomial with roots $m_1, \ldots, m_n$:

$$a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_m = 0$$
$$\Rightarrow \; a_0 (m - m_1)(m - m_2) \ldots (m - m_n) = 0$$

– *Case #1*: All roots are real.

$$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}.$$

– *Case #2*: Some roots are complex-conjugate pairs.
Consider the case of one pair with $m_1 = a + \imath b$ and $m_2 = a - \imath b$.

$$y(x) = [c_1 \cos(bx) + c_2 \sin(bx)] e^{ax} + \cdots .$$

– *Case #3*: Some roots a repeated [gex110].
Consider the case of real root m_1 occurring k times.

$$y(x) = [c_1 + c_2 x + \cdots + c_k x^{k-1}] e^{m_1 x} + \cdots .$$

Linearly damped harmonic oscillator:

The three cases identified in the previous section are all realized in this dynamical system.

System specifications: mass m, spring constant k, attenuation γ.

Equation of motion for position $x(t)$: $m\ddot{x} = -kx - \gamma\dot{x}$.

Damping parameter: $\beta \doteq \dfrac{\gamma}{2m}$.

Characteristic frequency: $\omega_0 \doteq \sqrt{\dfrac{k}{m}}$.

Linear homogeneous ODE: $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$.

Exponential ansatz: $x(t) = e^{rt}$.

Characteristic polynomial: $r^2 + 2\beta r + \omega_0^2 = 0$.

Roots: $r_\pm = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$.

Initial conditions: $x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0$.

Case #1: overdamped motion: $\Omega_1 \doteq \sqrt{\beta^2 - \omega_0^2} > 0$.

- Linearly independent solutions: $e^{r_+ t},\ e^{r_- t}$.
- General solution: $x(t) = \left(A_+ e^{\Omega_1 t} + A_- e^{-\Omega_1 t}\right) e^{-\beta t}$.
- Amplitudes: $A_+ = \dfrac{\dot{x}_0 - r_- x_0}{2\Omega_1}, \quad A_- = \dfrac{r_+ x_0 - \dot{x}_0}{2\Omega_1}$.

Case #2: underdamped motion: $\omega_1 \doteq \sqrt{\omega_0^2 - \beta^2} > 0$.

- Linearly independent solutions: $e^{r_+ t},\ e^{r_- t}$.
- General solution:

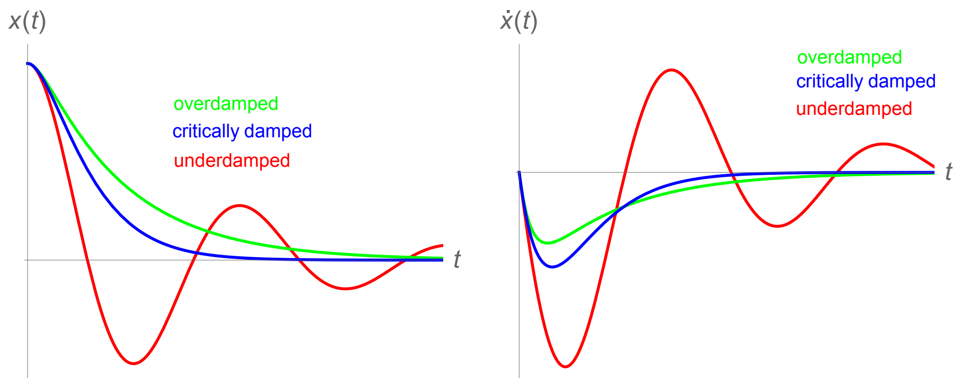
$$x(t) = (A \cos \omega_1 t + B \sin \omega_1 t)\, e^{-\beta t} = D \cos(\omega_1 t - \delta) e^{-\beta t}.$$

- Amplitudes and phase:

$$A = x_0, \quad B = \frac{\dot{x}_0 + \beta x_0}{\omega_1}, \quad D = \sqrt{A^2 + B^2}, \quad \delta = \arctan \frac{B}{A}.$$

Case #3: critically damped motion: $\omega_0^2 = \beta^2, \quad r = -\beta$.

- Linearly independent solutions: $e^{rt},\ te^{rt}$.
- General solution: $x(t) = (A_0 + A_1 t) e^{-\beta t}$.
- Amplitudes: $A_0 = x_0, \quad A_1 = \dot{x}_0 + \beta x_0$.



The plots for position [left] and velocity [right] pertain to initial conditions $x_0 = 1,\ \dot{x}_0 = 0$.

Particular solution of inhomogeneous ODE:

The presenc of an inhomogeneity $R(x)$ in a linear ODE with constant coefficients call for a particular solution $Y_\mathrm{p}(x)$ to be determined and added to the complementary solution $Y_\mathrm{c}(x)$ investigated earlier.

Here we discuss two methods. The second is a generalization of the first.

- The method of undetermined constant parameters is applicable if $R(x)$ is from a selective catalog of functions.
- The method of variation of parameters is applicable more generally for differentiable functions $R(x)$.

Method of undetermined constant parameters:

For inhomogeneity function $R(x)$ from the list on the left, the particular solution has a corresponding structure as shown in the list on the right for specific values (to be determined) of the constant parameters.

fe^{px}	ae^{px}
$f \cos px + g \sin px$	$a \cos px + b \sin px$
$f_0 x^k + f_1 x^{k-1} + \cdots + f_k$	$a_0 x^k + a_1 x^{k-1} + \cdots + a_k$
$e^{qx}(f \cos px + g \sin qx)$	$e^{qx}(a \cos px + b \sin px)$
$e^{qx}(f_0 x^k + f_1 x^{k-1} + \cdots + f_k)$	$e^{qx}(a_0 x^k + a_1 x^{k-1} + \cdots + a_k)$
$(f_0 x^k + \cdots + f_k) \cos px + (g_0 x^k + \cdots + g_k) \sin px$	$(a_0 x^k + \cdots + a_k) \cos px + (b_0 x^k + \cdots + b_k) \sin px$
$e^{qx}(f_0 x^k + \cdots + f_k) \cos px + e^{qx}(g_0 x^k + \cdots + g_k) \sin px$	$e^{qx}(a_0 x^k + \cdots + a_k) \cos px + e^{qx}(b_0 x^k + \cdots + b_k) \sin px$

[image from Spiegel 1971]

If $R(x)$ is a linear combination of entries shown on the left, then the trial function for the particular solution is a corresponding linear combination of entries on the right.

If a term suggested by the appropriate entry on the right already appears in $Y_\mathrm{c}(x)$, then that term must be multiplied by powers of the independent variable x until it becomes distinctive (for reasons explored in [gex110]).

Method of variation of parameters: