ODEs of Classical Dynamical Systems [gam3]

Newton's equation of motion for an autonomous system of one degree of freedom is a 2nd-order ODE of the form,

$$m\ddot{x} = F(x, \dot{x}), \tag{1}$$

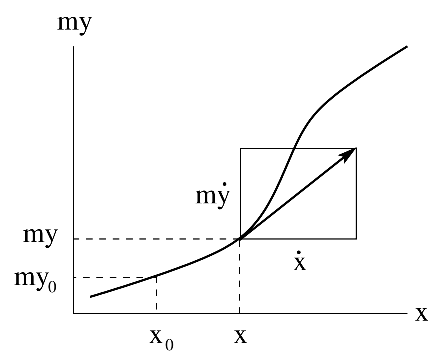
where the acting force $F(x, \dot{x})$ depends, in general, on position and velocity.[1]

Flow in phase plane:

Equivalent to the 2nd-order ODE (1) is the pair of 1st-order ODEs,

$$\dot{x} = y, \qquad \dot{y} = F(x, y)/m, \tag{2}$$

a representation central to Hamiltonian mechanics (see PHY520). They represent a flow in the phase-plane (x, p), where $p = m\dot{x}$ is the momentum.



– The graph shows (schematically) a solution of (2), named phase-space trajectory, for initial conditions (x_0, y_0).

– Also shown is the tangent vector to the trajectory at the point (x, my), realized at some later time.

– The components $(\dot{x}, m\dot{y})$ of the tangent vector depend on the location (x, y) as determined by Eqs. (2).

– The vectors with components $\dot{x} = y$, $m\dot{y} = F(x, y)$ thus specify a vector field, available prior to the solution of the ODEs (2).

– Any particular solution of Eqs. (2) is a curve which is tangential to this vector field at all points in the phase plane.

– The structure of the vector field holds key clues about the structure of the trajectories.

[1]Non-autonomous systems, with an acting force $F(x, \dot{x}, t)$, can be interpreted as autonomous systems of two degrees of freedom.

In the following, our focus is on the variety of structure in the solutions of systems of 1st-order ODEs of two or more dynamical variables.

Dynamical systems of two variables:

A generic classical dynamical system of two independent variable can expressed in a pair of 1st-order ODEs as follows:

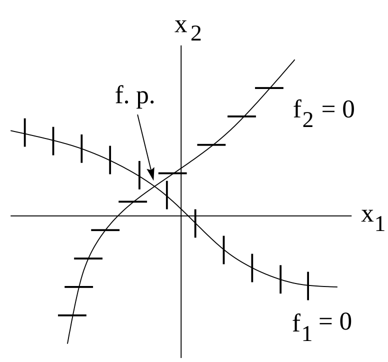
$$\dot{x}_1 = f_1(x_1, x_2), \qquad \dot{x}_2 = f_2(x_1, x_2). \tag{3}$$

- Equations (3) determine a vector field $(\dot{x}_1, \dot{x}_2)$ in the plane (x_1, x_2).
- Any particular solution of (3) describes a unique trajectory.
- Trajectories are tangential to the vector field everywhere.
- Trajectories do not intersect themselves or each other. Solutions at all points must have a unique slope if it is nonzero.
- The trajectories describe a flow in the (x_1, x_2)-plane with the tangent vectors $(\dot{x}_1, \dot{x}_2)$ in the role of a velocity field.
- The non-intersection rule is highly constraining in two dimensions. Think of road traffic with no intersections, no tunnels, and no bridges.

Isoclines:

Isoclines are sets of curves on which all trajectories have tangents with equal direction.

- Isoclines intersected *vertically* by all trajectories are determined by the curves representing $f_1(x_1, x_2) = 0$.
- Isoclines intersected *horizontally* by all trajectories are determined by the curves representing $f_2(x_1, x_2) = 0$.



All points of intersection between a curve of vertical isocline and a curve of horizontal isocline are fixed points, where $\dot{x}_1 = 0$, $\dot{x}_2 = 0$.

Fixed points:

Fixed points in the flow described by ODEs (3) are special trajectories for which the system is and remains at rest. The tangent vector $(\dot{x}_1, \dot{x}_2)$ vanishes.

Fixed points are landmarks, which govern the nature of the flow in their vicinity. The classification of fixed points in a 2D flow proceeds as follow.

– Equation of motion: $\dot{x} = f(x,y), \quad \dot{y} = g(x,y)$.

– Tangent vector field: $\mathbf{v}(x,y) = (\dot{x}, \dot{y}) = \big(f(x,y),\ g(x,y)\big)$.

– Fixed point locations: $\mathbf{v}(x_k, y_k) = 0$ from $f(x,y) = g(x,y) = 0$.

– Tangent vector field linearized around fixed point at $\mathbf{r}_k = (x_k, y_k)$:

$$\mathbf{v} = \mathbf{A}_k \cdot (\mathbf{r} - \mathbf{r}_k) + \mathrm{O}(\mathbf{r} - \mathbf{r}_k)^2. \tag{4}$$

– Jacobian matrix evaluated at fixed point k:

$$\mathbf{A}_k = \left(\begin{array}{cc} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{array} \right)_{\mathbf{r}_k} = \left(\begin{array}{cc} a_k & b_k \\ c_k & d_k \end{array} \right), \tag{5}$$

where a_k, b_k, c_k, d_k are, in general, real numbers. They characterize the nature of the fixed point k.

– The eigenvalues of a 2×2 matrix depends only on its trace and its determinant:

  ▷ Jacobian matrix: $\mathbf{A} = \left(\begin{array}{cc} a & b \\ c & d \end{array} \right)$,

  ▷ trace: $\tau = a + d$,

  ▷ determinant: $\delta = ad - bc$,

  ▷ characteristic equation:

$$|\mathbf{A} - \lambda \mathbf{I}| = \left| \begin{array}{cc} a - \lambda & b \\ c & d - \lambda \end{array} \right| = \lambda^2 - \tau\lambda + \delta = 0, \tag{6}$$

  ▷ eigenvalues: $\lambda = \dfrac{\tau}{2} \pm \sqrt{\dfrac{\tau^2}{4} - \delta}$.

– There are ten types of fixed points, divided into three groups:

#1: $\tau^2 > 4\delta \quad \Rightarrow \lambda_1 \neq \lambda_2$, real (nodes, hyperbolic point).

#2: $\tau^2 < 4\delta \quad \Rightarrow \lambda_1 = \lambda_2^*$, complex (spirals, elliptic point).

#3: $\tau^2 = 4\delta \quad \Rightarrow \lambda_1 = \lambda_2$, real (stars, improper nodes).

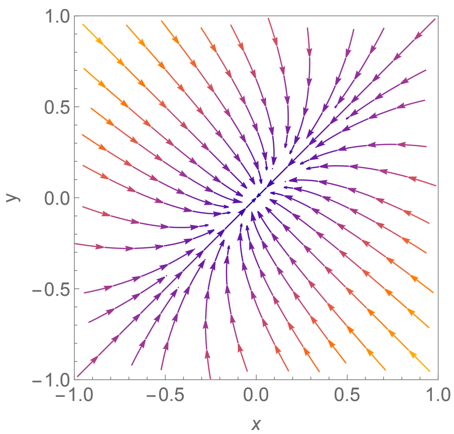
– Nodes:

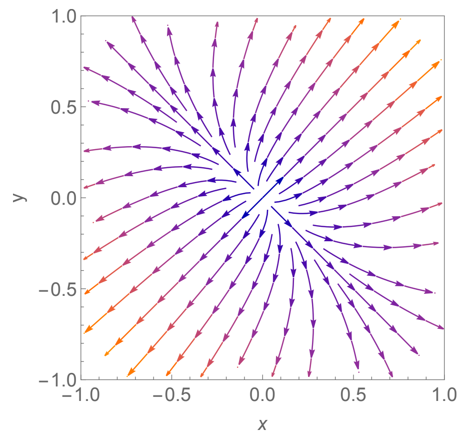
▷ stable node (attractor): $\lambda_2 < \lambda_1 < 0$.

$$\begin{aligned} &a = -1,\ b = 0.4,\ c = 0.4,\ d = -1 \\ \Rightarrow\ &\tau = -2,\ \delta = 0.84,\ \lambda_1 = -0.6,\ \lambda_2 = -1.4. \end{aligned}$$

▷ unstable mode (repellor): $0 < \lambda_2 < \lambda_1$.

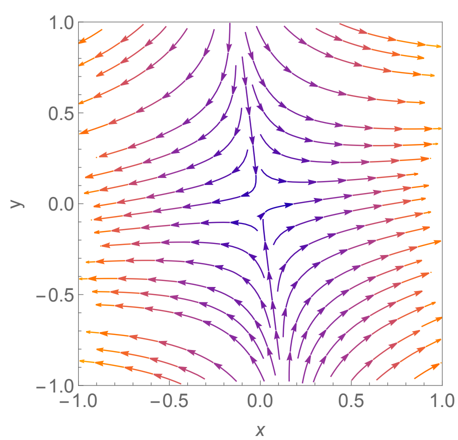
$$\begin{aligned} &a = 1,\ b = 0.4,\ c = 0.4,\ d = 1 \\ \Rightarrow\ &\tau = 22,\ \delta = 0.84,\ \lambda_1 = 1.4,\ \lambda_2 = 0.6. \end{aligned}$$





– Hyperbolic point: $\lambda_2 < 0 < \lambda_1$.

$$\begin{aligned} &a = 1.5,\ b = 0.25,\ c = 0.25,\ d = -0.5 \\ \Rightarrow\ &\tau = 1,\ \delta \simeq -0.813\ldots,\ \lambda_1 \simeq 1.531,\ \lambda_2 \simeq -0.531. \end{aligned}$$



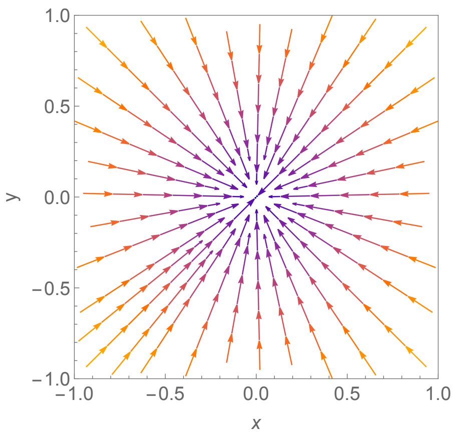
– Stars:

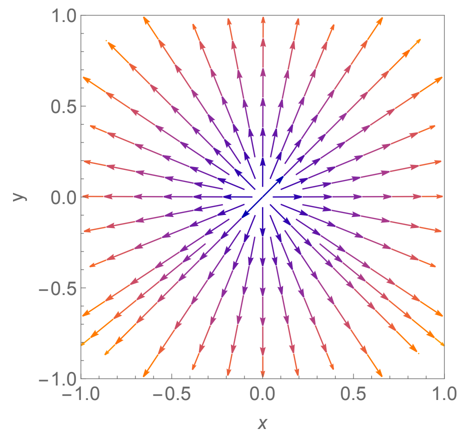
▷ stable star (attractor): $\lambda_2 = \lambda_1 < 0,\ b = c = 0.$

$$a = -1,\ b = 0,\ c = 0,\ d = -1$$
$$\Rightarrow\ \tau = -2,\ \delta = 1,\ \lambda_1 = -1,\ \lambda_2 = -1.$$

▷ unstable star (repellor): $0 < \lambda_2 = \lambda_1,\ b = c = 0.$

$$a = 1,\ b = 0,\ c = 0,\ d = 1 \quad \Rightarrow \quad \tau = 2,\ \delta = 1,\ \lambda_1 = 1,\ \lambda_2 = 1.$$





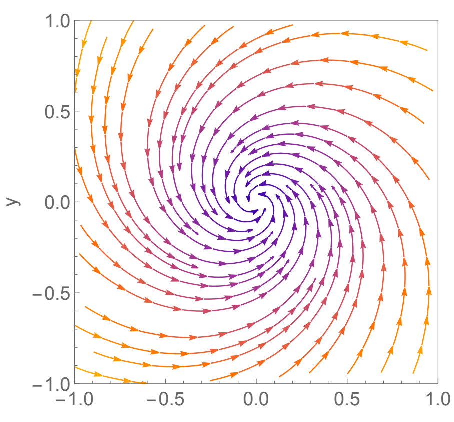
– Spirals:

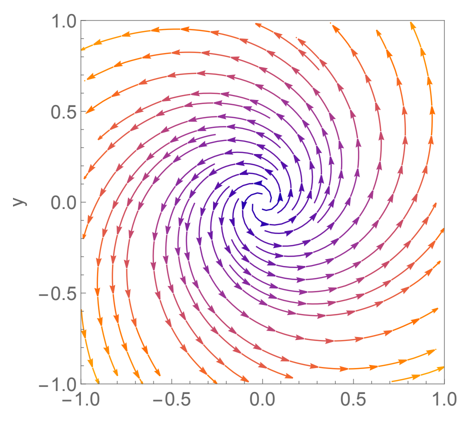
▷ stable spiral (attractor): $\lambda_1 = \bar{\lambda}_2 = \alpha + \imath\omega,\ \alpha < 0.$

$$a = -0.5,\ b = -1,\ c = 1,\ d = -0.5$$
$$\Rightarrow\ \tau = -1,\ \delta = 1.25,\ \lambda_1 = -0.5 + \imath,\ \lambda_2 = -0.5 - \imath.$$

▷ unstable spiral (repellor): $\lambda_1 = \bar{\lambda}_2 = \alpha + \imath\omega,\ \alpha > 0.$

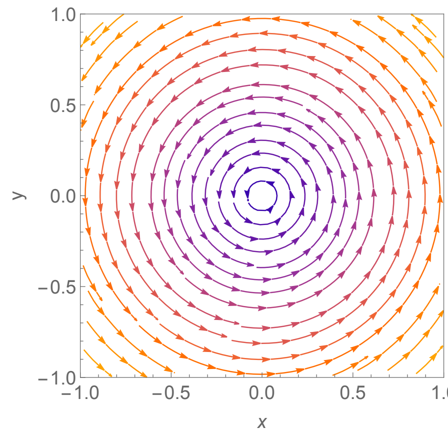
$$a = 0.5,\ b = -1,\ c = 1,\ d = 0.5$$
$$\Rightarrow\ \tau = 1,\ \delta = 1.25,\ \lambda_1 = 0.5 + \imath,\ \lambda_2 = 0.5 - \imath.$$





– Elliptic point: $\lambda_1 = \bar{\lambda}_2 = \imath\omega$.

$$a = 0,\ b = -1,\ c = 1,\ d = 0$$
$$\Rightarrow\ \tau = 0,\ \delta = 1,\ \lambda_1 = \imath,\ \lambda_2 = -\imath.$$



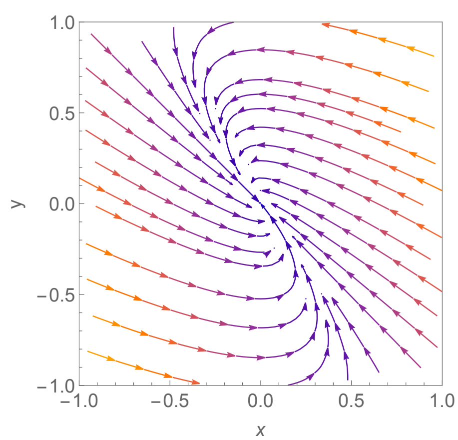
– Improper nodes:

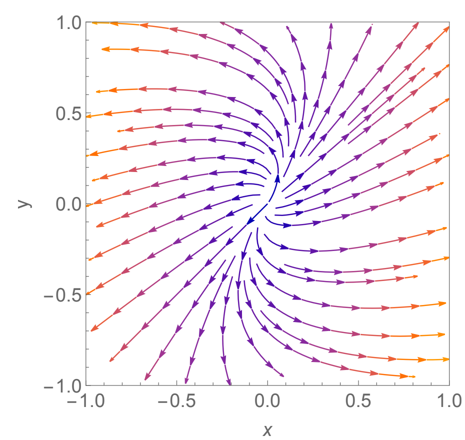
▷ stable improper node (attractor): $\lambda_1 = \lambda_2 < 0,\ b \neq 0$ or $c \neq 0$.

$$a = -2,\ b = -1,\ c = 1,\ d = -0$$
$$\Rightarrow\ \tau = -2,\ \delta = 1,\ \lambda_1 = -1,\ \lambda_2 = -1.$$

▷ unstable improper node (repellor): $0 < \lambda_2 = \lambda_1,\ b \neq 0$ or $c \neq 0$.

$$a = 1.5,\ b = -0.5,\ c = 0.5,\ d = 0.5$$
$$\Rightarrow\ \tau = 2,\ \delta = 1,\ \lambda_1 = 1,\ \lambda_2 = 1.$$





Conservative forces:

Forces operating on dynamical systems with one degree of freedom are guaranteed to be conservative if they only depend on position (not on velocity). This eliminates all but types of fixed points.

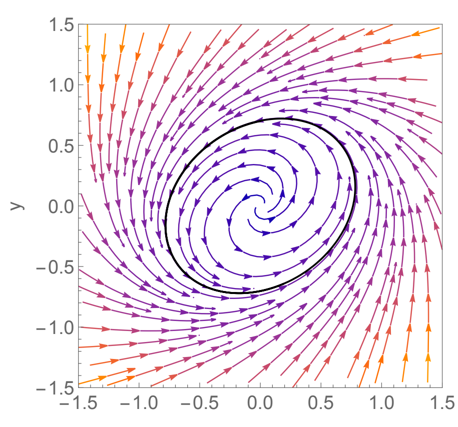
- Newton's equation of motion: $m\ddot{x} = F(x)$.
- Equivalent 1^{st}-order ODEs: $\dot{x} = y,\ \dot{y} = F(x)/m \doteq f(x)$.
- Jacobian matrix: $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ \partial f/\partial x & 0 \end{pmatrix}$.
- Trace: $\tau = 0$.
- The condition $\tau = 0$ can be satisfied by elliptic and hyperbolic points.
- All other fixed points are either attractors or repellors and have $\tau \neq 0$.
- The flow of conservative systems is area-preserving (volume-preserving in systems with more degrees of freedom. This a central topic in Hamiltonian mechanics.
- Attractors and repellors signal the presence of dissipative effects (friction, damping, attenuation).

Limit cycle:

Not all attractors are fixed points. For dynamical systems with two variables there is also the limit cycle.

This system has a spiral repellor at $(x = 0, y = 0)$ and an oval-shaped limit cycle to which trajectories gravitate from inside and from outside [gex106].

$$\dot{x} = x\left(1 - \sqrt{x^2 + y^2}\right) - y, \quad \dot{y} = y\left(\tfrac{1}{2} - \sqrt{x^2 + y^2}\right) + x, \tag{7}$$



Dynamical systems of three variables:

Solutions of dynamical systems characterized by a 3rd-order ODE or, equivalently, three coupled 1st-order ODEs, for example,

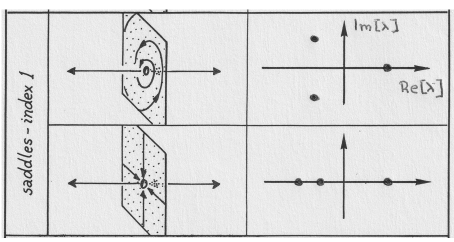
$$\dot{x} = f(x, y, z), \quad \dot{y} = g(x, y, z), \quad \dot{z} = h(x, y, z), \tag{8}$$

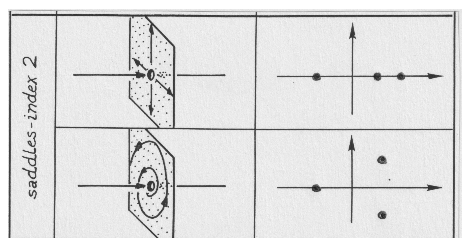
allow much higher complexity. These solutions can be visualized as trajectories in (x, y, z)-space which must not intersect themselves or each other.

The non-intersection condition is, of course, much less constraining in 3D than in 2D, which opens the gate for qualitatively new phenomena.

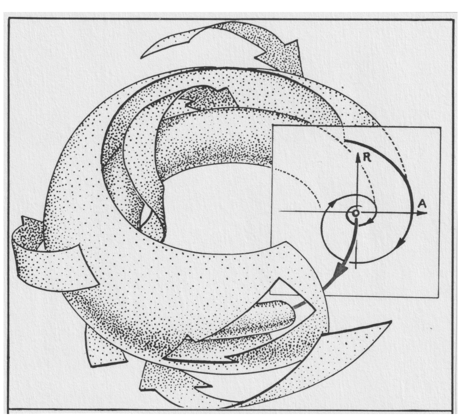
Expected generalizations are invariant structures of dimensions $D = 0, 1, 2$.

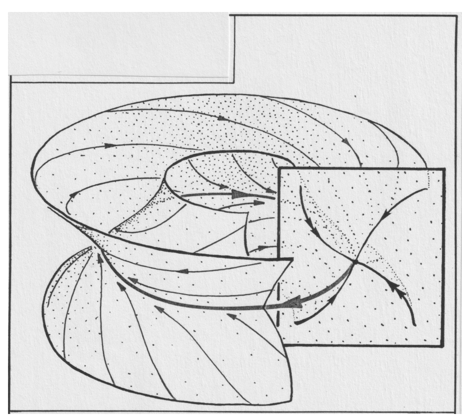
- Invariant structures of dimension D are manifolds to which solutions are confined for all times if they start on the manifold. Some invariant structures are attractors, others are repellors or neither.
- Fixed points are invariant structures of dimension $D = 0$. The list of types is larger. The types depend on attributes of three eigenvalues of the Jacobian matrix. Noteworthy additions are saddle points.



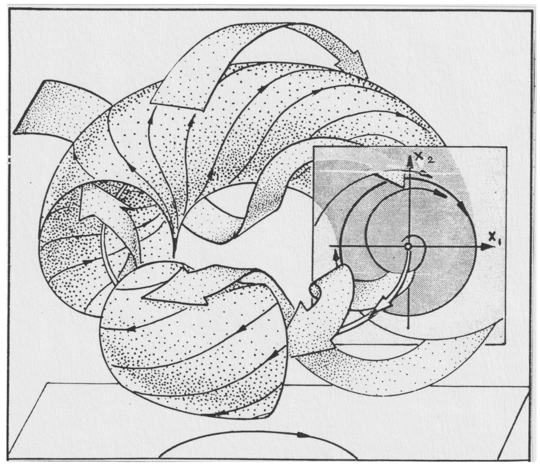


- Among higher-dimensional invariant structures we focus on attractors. Limit cycles are 1D attractors. There are now variations in the mode of attraction. This includes spiral cycles (left) and nodal cycles (right).





– The most common 2D attractor is a torus. Trajectories on the torus must not intersect. Rational tori contain infinitely many periodic trajectories, irrational tori just one aperiodic trajectory.



[images from Abraham and Shaw 1984]

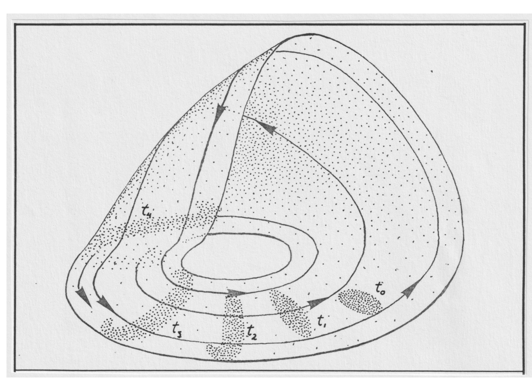
Unexpected generalizations include strange attractors of fractal dimension. They are invariant structures of infinite area, but zero volume.

One prominent example is the Rössler band, which shows up as a strange attractor upon integration of the set of ODEs,

$$\dot{x} = -y - z, \quad \dot{y} = x + \frac{1}{5}\,y, \quad \dot{z} = \frac{1}{5} + z(x - 5.7). \tag{9}$$

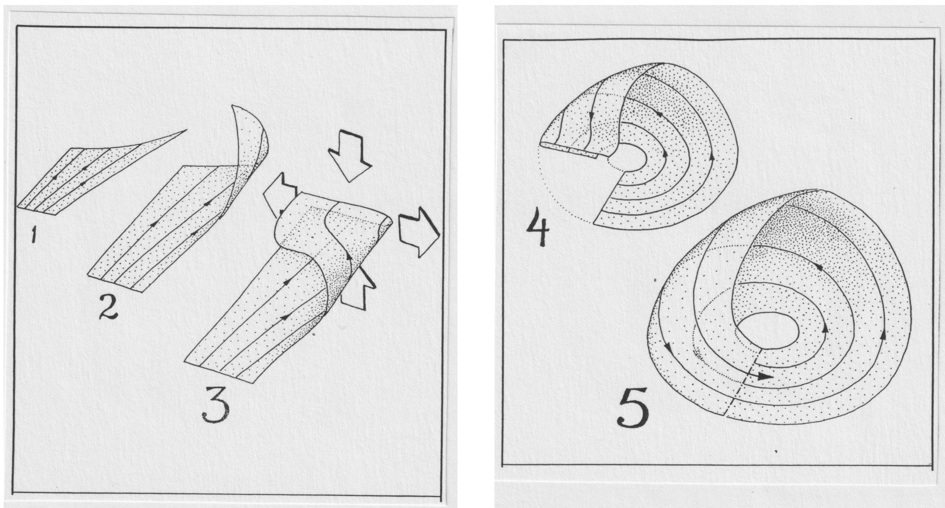
It's the nonlinearity in the third ODE which makes all the difference.

The Rössler band (sketched below) is a sort of infinitely long ribbon that grows wider as time evolves, bends into the original width, and folds into a stack of zero height.



The continual stretching, bending, and stacking process in its consecutive stages is depicted below.

Trajectories with nearby initial conditions diverge exponentially fast during stretching. The bending and stacking rings them close to each other again, but with rapidly diminishing correlation.



[images from Abraham and Shaw 1984]

Is classical mechanics a deterministic theory?

The affirmative answer of 18^{th}-century scholars was largely accepted until it was undermined by the discovery of deterministic chaos around 1900.

- The laws of classical mechanics come in the form of ODEs. Solutions of ODEs with initial conditions fully specified are unique.
- The future and the past are determined if the presence is known. More precisely: fully determined if fully known.
- Alas, fully known is not in the cards. Does that make a difference? The answer depends on how solutions with nearby initial conditions evolve in time relative to one another.
- In regular solutions, the uncertainty about initial conditions propagates as a power of the time evolved. In irregular (chaotic) solutions, the uncertainty propagates exponentially in time.
- The tiniest amount of uncertainty affects the predictability of the future in the two kinds of solutions very differently.

- For the examination of this difference we compare the information (in bytes) contained (i) in the initial conditions and the ODE and (ii) in a data set of the trajectory over a time interval Δt.
  - ▷ A regular trajectory takes M_0 bytes of programming and $a \ln \Delta t$ bytes of initial conditions ($a > 0$, a constant) to calculate the trajectory within given precision over the time interval Δt.
  - ▷ An irregular trajectory takes M_0 bytes of programming and $b\Delta t$ bytes of initial conditions ($b > 0$, a constant) to calculate the trajectory within given precision over the time interval Δt.
  - ▷ The difference in the second contribution, $a \ln \Delta t$ vs $b\Delta t$, is due to the different error propagation: power law vs exponential.
  - ▷ For both types of trajectories, mapping out the trajectory takes $c\Delta t$ bytes ($c > 0$, a constant).
  - ▷ The claim of classical mechanics for being deterministic hinges on the ratio of the bytes needed to compute a trajectory from an ODE and the bytes needed to map out that trajectory.
  - ▷ Regular trajectory: $\lim_{\Delta t \to \infty} \dfrac{M_0 + a \ln \Delta t}{c\Delta t} = 0.$
  - ▷ Irregular trajectory: $\lim_{\Delta t \to \infty} \dfrac{M_0 + b\Delta t}{c\Delta t} = \dfrac{b}{c} > 0.$
  - ▷ If that ratio approaches zero in the long-time limit, as it does for regular trajectories, then the ODE plus initial condition has predictive power that that is consistent with a claim for determinism.
  - ▷ If the ratio stays nonzero, as it does for irregular trajectories, then no such claim holds up. Calculating a trajectory has no advantage over mapping it out from observational data for long times.
- Soon after classical chaos was recognized to undermine claims for determinism by classical mechanics, quantum mechanics undermined claims for determinism from an entirely different side.