Summation Convention [gam2]

Expressions in tensor analysis teem with indices in the form of subscripts and superscripts and often include sums over some of the indices.

Einstein recognized that the prevailing patterns in tensor expressions make it possible to simplify the notation without causing ambiguity or confusion. His summation convention is now adopted wherever tensors are ubiquitous.

- In tensor expressions, identical indices appear either once or twice.
- The tacit assumption is that all indices have the same range, for example, $i = 1, 2, \ldots, n$.
- Free indices occur once and have a fixed (albeit variable) value,
- Dummy indices occur twice and are summed over.
- In the equation below, i is a free index and j a dummy index. The summation convention is adopted in the second expression:

$$y_i = \sum_{j=1}^{n} a_{ij}x_j \to a_{ij}x_j \quad : \; i = 1, \ldots, n.$$

- In the sum of two tensor expressions, the dummy indices of each term can be changed independently:

$$a_{ij}x_j + b_{ij}x_j = a_{ij}x_j + b_{ik}x_k.$$

- Some tensor expressions must be rewritten to fit the pattern:

$$\begin{aligned} &a_{ij}x_jx_j \to a_{ij}(x_j)^2, \\ &a_{ij}(x_j + y_j) \to a_{ij}z_j \text{ with } z_j \doteq x_j + y_y. \end{aligned}$$

- The first expression below involves a single sum, the second expression a double sum. The sequence of summation does not matter.

$$a_{ij}x_j, \quad a_{ij}x_iy_j.$$

- When one tensor expression is substituted in another tensor expressions, it may be necessary to rename some of the indices.
  - ▷ Example: $b_{ij}y_ix_j$ (two sums) with $y_i = a_{ij}x_j$ (one sum).
  - ▷ Rewrite: $y_i = a_{ik}x_k$.
  - ▷ Substitute: $b_{ij}(a_{ik}x_k)x_j$.
  - ▷ Rearrange: $a_{ik}b_{ij}x_kx_j$ (three sums).

- Recognize which expressions are equal and which are not:

$$a_{ij}x_j = a_{ik}x_k, \quad a_{ij}x_j \neq a_{kj}x_j.$$

- The same rules hold if sum or all of the indices are superscripts.
- The sequence of indices does matter, in general. The notation (for mixed indices) must make that clear. Write ${a^i}_j$ or ${a_j}^i$.
- Kronecker delta: $\delta_{ij} = \delta^{ij} = {\delta_i}^j = {\delta^i}_j = \begin{cases} 1 & : \ i = j \\ 0 & : \ i \neq j \end{cases}.$
  - ▷ $x_i x_j \delta_{ij} = x_i x_i$,
  - ▷ $a_{ij} x_i x_j \delta_{ij} = a_{ii}(x_i)^2 \to b_i(x_i)^2, \quad b_i = a_{ii}.$

  In both applications a double sum is reduced to a single sum.