Conjugate Functions in Electrostatics [gam1]

The electrostatic potential $\Phi(x, y)$ in the space outside a configuration of conductors with continuous translational symmetry (in z-direction) is a solution of the Laplace equation,

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0.$$

The attributes of analytic functions and harmonic functions as discussed in [gmd7] can be exploited to find solutions of electrostatic problems.

Pick an analytic function $f(z) = g(x, y) + \imath h(x, y)$ of which either $g(x, y)$ or $h(x, y)$ satisfies the boundary conditions of a given electrostatic problem.

Both functions are guaranteed to be harmonic, i.e. to satisfy the Laplace equation separately:

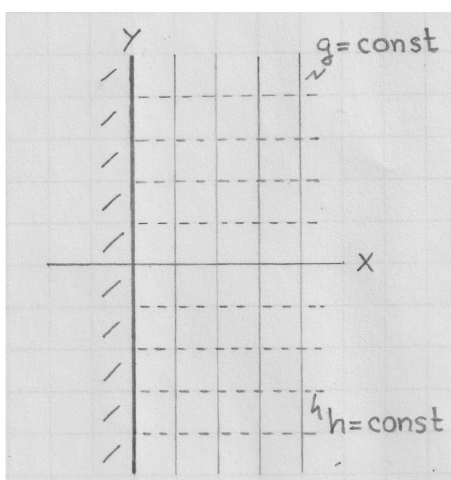
$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 0, \quad \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0.$$

If it is $g(x, y)$ which satisfies the boundary condition we set $\Phi(x, y) = g(x, y)$. Equipotential lines are then represented by curves $g(x, y) = \text{const}$.

All curves $h(x, y) = \text{const}$ are orthogonal to equipotential lines at points of intersection [gmd7]. Therefore, they represent electrostatic field lines.

The role of the functions g and h is interchanged if it $h(x, y)$ which satisfies the electrostatic boundary conditions.

Application to plane surface with charge density σ on yz-plane ($x < 0$ is inside the conductor and $x > 0$ outside):



– Analytic function: $F(z) = cz = cx + \imath cy$.

– Harmonic functions: $g = cx, \quad h = cy$.

– Analyticity check: $\dfrac{\partial g}{\partial x} = \dfrac{\partial h}{\partial y} = c, \quad \dfrac{\partial g}{\partial y} = -\dfrac{\partial h}{\partial x} = 0$.

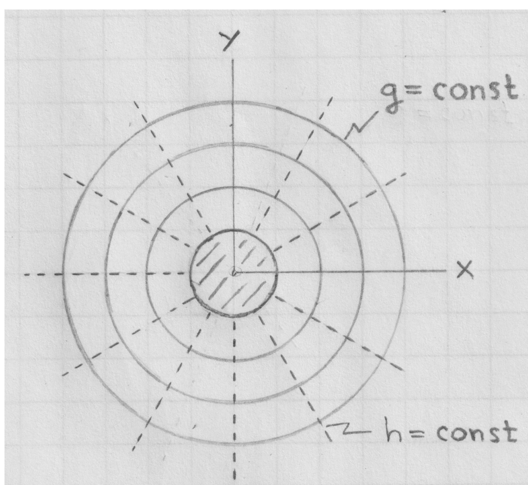
– Equipotential lines: $g = \text{const} \quad \Rightarrow \; x = \text{const}$.

– Field lines: $h = \text{const} \quad \Rightarrow \; y = \text{const}$.

– Electric field: $\mathbf{E} = \dfrac{\sigma}{\epsilon_0}\,\hat{\mathbf{i}} = -\nabla\Phi$.

– Electric potential: $\Phi = -\dfrac{\sigma}{\epsilon_0}\,x$.

Application to charged cylindrical surface of radius r_0 centered at z-axis with charge density σ (shown in cross section).



– Analytic function: $F(z) = c\ln z = c\ln r + \imath c\phi, \quad z = re^{\imath\phi}$.

– Real part: $g = c\ln r, \quad r = \sqrt{x^2+y^2}$.

– Imaginary part: $h = c\phi, \quad \phi = \arctan\dfrac{y}{x}$.

– Analyticity check: $\dfrac{\partial g}{\partial x} = \dfrac{\partial h}{\partial y}, \quad \dfrac{\partial g}{\partial y} = -\dfrac{\partial h}{\partial x}$.

$$\frac{\partial g}{\partial x} = \frac{dg}{dr}\frac{\partial r}{\partial x} = \frac{cx}{x^2+y^2}, \quad \frac{\partial h}{\partial y} = \frac{dh}{d\phi}\frac{\partial \phi}{\partial y} = \frac{cx}{x^2+y^2}.$$

$$\frac{\partial g}{\partial y} = \frac{dg}{dr}\frac{\partial r}{\partial y} = \frac{cy}{x^2+y^2}, \quad \frac{\partial h}{\partial x} = \frac{dh}{d\phi}\frac{\partial \phi}{\partial x} = -\frac{cy}{x^2+y^2}.$$

- Equipotential lines: $g = \text{const} \quad \Rightarrow \; r = \text{const} \quad$ (concentric circles).
- Field lines: $h = \text{const} \quad \Rightarrow \; \phi = \text{const} \quad$ (radial lines).
- Electric potential: $\Phi = \Phi_0 \ln(r/r_0)$.
- Electric field: $\mathbf{E} = -\nabla\Phi = -\dfrac{\Phi_0}{r}\hat{\mathbf{r}} = \dfrac{\lambda}{2\pi\epsilon_0 r}\hat{\mathbf{r}} = \dfrac{\sigma r_0}{\epsilon_0 r}\hat{\mathbf{r}}, \quad \lambda \doteq 2\pi r_0 \sigma$.
- Field strength at $r = r_0$: $E_0 = \dfrac{\sigma}{\epsilon_0} = -\dfrac{\Phi_0}{r_0} \quad \Rightarrow \; \Phi_0 = -\dfrac{\sigma r_0}{\epsilon_0}$.

Further applications:

- Conducting plates intersecting at right angle [gex57].
- Electric potential and field at edge of large conducting plate [gex58].
- Fringe electric potential and field of parallel plates [gex59].