[lex9] Electric dipole field

Given the dipole term in the multipole expansion of the electric potential $\Phi(\mathbf{x})$, derive the following expression for the electric dipole field $\mathbf{E}(\mathbf{x})$:

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}, \quad \Rightarrow \ \mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\frac{3\hat{\mathbf{r}}(\mathbf{p}\cdot\hat{\mathbf{r}}) - \mathbf{p}}{r^3},$$

where $\mathbf{p}$ is the electric dipole moment and $\hat{\mathbf{r}} = \mathbf{x}/r$ the unit vector pointing from the dipole (assumed localized) to the field point.
(a) Work out the solution by hand.
(b) Work out the solution in a Mathematica notebook. A vector is List. Its magnitude is Norm. The dot product is Dot, The gradient is Grad.
(c) What is the relative orientation of $\mathbf{E}$ and $\mathbf{p}$ (i) if $\hat{\mathbf{r}}$ and $\mathbf{p}$ are parallel, (ii) if $\hat{\mathbf{r}}$ and $\mathbf{p}$ are perpendicular, (iii) if $\hat{\mathbf{r}}$ and $\mathbf{p}$ are antiparallel.

Solution: