[lex83] Maxwell's equations for the scalar and vector potentials

Starting from Maxwell's equation for the electric and magnetic fields,

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t},$$

and using the relations between fields and potentials,

$$\mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A},$$

(a) derive the following Maxwell's equations for the scalar and vector potential,

$$\nabla^2 \Phi + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho}{\epsilon_0}, \quad \nabla^2 \mathbf{A} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla \left(\nabla \cdot \mathbf{A} + \mu_0 \epsilon_0 \frac{\partial \Phi}{\partial t} \right) = -\mu_0 \mathbf{J},$$

(b) demonstrate the invariance of these equations under the gauge transformation,

$$\mathbf{A}' = \mathbf{A} + \nabla f, \quad \Phi' = \Phi - \frac{\partial f}{\partial t}.$$

Solution: