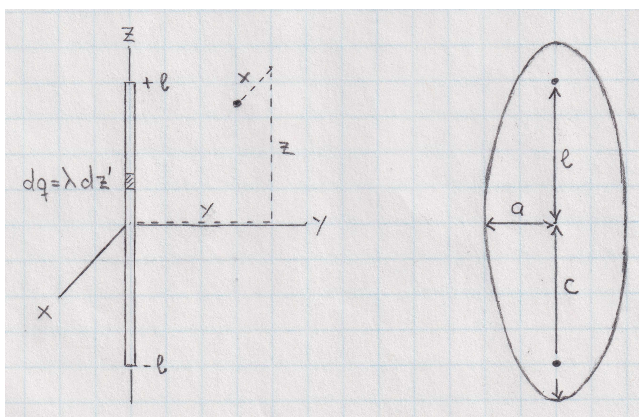
[lex42] Electric potential of charged rod

Consider a uniformly charged rod of length $2l$ and with linear charge density λ positioned on the z-axis and centered at the xy-plane.
(a) Show that the electric potential is

$$\Phi(x,y,z) = k\lambda \ln\left(\frac{z+l+\sqrt{x^2+y^2+(z+l)^2}}{z-l+\sqrt{x^2+y^2+(z-l)^2}}\right), \quad k \doteq \frac{1}{4\pi\epsilon_0}$$

(b) Produce a contour plot of $\Phi/k\lambda$ for points in the (y,z)-plane with range $-2 \leq y \leq 2$, $-2 \leq z \leq 2$ using $l=1$. All distances are in arbitrary units.
(c) The contours look like ellipses. Are they? With focal points at the end of the rod? Let's check that out numerically first. Superimpose the contour plot with a model ellipse, $y^2/a^2 + z^2/c^2 = 1$ with semimajor axis c and semiminor axis $a = \sqrt{c^2 - l^2}$. Choose $l = 1$ and $c = 1.5$ for that purpose.
(d) Produce a stream plot for the electric field $\mathbf{E}(x,y,z) = -\nabla\Phi(x,y,z)$, again for points in the (y,z)-plane with the same range.
(e) Verify analytically that the contours are indeed ellipses by showing that if we substitute the relation $y^2/a^2 + z^2/c^2 = 1$ with $a = \sqrt{c^2 - l^2}$ in the expression $\Phi(0,y,z)$ we end up with a constant, namely an equipotential line. Determine that constant, parametrized by the semimajor axis c.

Hints: Use Mathematica or equivalent for parts (a)-(d). Part (a) reduces to a definite integral. For parts (b)-(d) the Mathematica commands ContourPlot, ParametricPlot, Grad, and StreamPlot have proven useful.



Solution: