[lex27] Electric potential and field at edge of large conducting plate

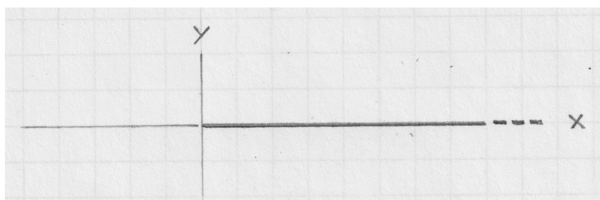
A large and thin conducting plate is positioned in the horizontal plane at $x \geq 0$ as shown in cross section. Far away from the edge at $x = 0$, the charge on both surfaces of the plate is known to be uniformly distributed and to produce a uniform electric field in the vicinity: $E_y = (\sigma/\epsilon_0)\mathrm{sgn}(y)$. Near the edge, the charge distribution is non-uniform, described by a surface charge density $\sigma(x)$ and the electric field is non-uniform in direction and magnitude: $\mathbf{E} = E_x(x,y)\,\hat{\mathbf{i}} + E_y(x,y)\,\hat{\mathbf{j}}$. Use the method of conjugate functions from [lln7] for the analysis of this situation, employing the complex function,

$$F(z) \doteq A\sqrt{z} = g(x,y) + \imath h(x,y), \quad z \doteq x + \imath y.$$

(a) Show that the real and imaginary parts of this complex function are

$$g(x,y) = \frac{A}{\sqrt{2}}\left[\sqrt{x^2+y^2}+x\right]^{1/2}, \quad h(x,y) = \Phi(x,y) = \frac{A}{\sqrt{2}}\left[\sqrt{x^2+y^2}-x\right]^{1/2},$$

respectively, and that they satisfy the Cauchy-Riemann conditions, which makes them conjugate functions and solutions of the Laplace equation.
(b) One of the two functions, when equated with the electric potential $\Phi(x,y)$ satisfies the boundary condition, $\Phi(x,0) = 0$ for $x > 0$. Which is it?
(c) Design graphical representations of equipotential lines potential and field lines, which intersect orthogonally from the relations $g(x,y) = \text{const}$ and $h(x,y) = \text{const}$.
(d) From the gradient of the function $\Phi(x,y)$ evaluated at $x > 0$ and $y = 0$ derive the function $\sigma(x)$ representing the surface charge density using the local relation $E = \sigma/\epsilon$.



Solution: