## [lex24] Series expansion of off-center Coulomb potential

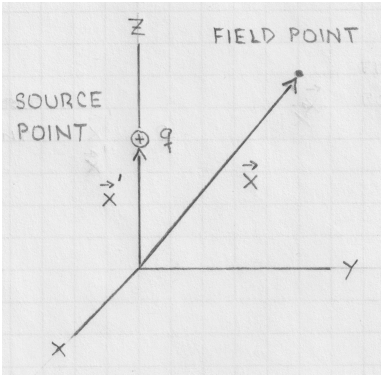
A point charge $q$ is positioned off center at $\mathbf{x}' = (r', 0, 0)$ in a system of spherical coordinates. The electric potential at point $\mathbf{x} = (r, \theta, \phi)$ is

$$\Phi(\mathbf{x}) = \frac{q}{4\pi\epsilon_0} \frac{1}{|\mathbf{x} - \mathbf{x}'|}. \tag{1}$$

It will prove useful in a number of contexts, including that of Green's functions, to express (1) as an expansion in the coordinates $r$ and $\theta$, specifically in the form,

$$V(\mathbf{x}) = V(r, \theta) = \frac{q}{4\pi\epsilon_0} \sum_{l=0}^{\infty} R_l(r) P_l(\cos\theta). \tag{2}$$

Use the generating function of Legendre polynomials [lln8] to find the functions $R_l(r)$ for $r \leq r'$ and for $r \geq r'$ in expression (2). The absence of the angle $\phi$ in this expansion is due to azimuthal symmetry of (1). The series is convergent everywhere except at the position of $q$.




**Solution:**