[lex185] Coherent state of quantum harmonic oscillator III

The pure one-parameter quantum state,

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle,$$

where α is the complex-valued parameter, is known as a coherent state of the quantum harmonic oscillator used for modeling a stream of photons [lln26].
(a) Show that the following overlap relation between coherent sates holds:

$$|\langle\beta|\alpha\rangle|^2 = e^{-|\alpha-\beta|^2}.$$

(b) Show that the following integrals over the unrestricted range of a complex parameter hold, where $\mathcal{I}$ is the identity operator:

$$\int d^2\beta \, |\langle\beta|\alpha\rangle|^2 = \pi, \quad \mathcal{I} = \frac{1}{\pi} \int d^2\alpha \, |\alpha\rangle\langle\alpha|.$$

(c) Prove the following results for the mean position and momentum in a coherent state:

$$\langle\alpha|\mathcal{Q}|\alpha\rangle = \sqrt{\frac{2\hbar}{m\omega}} \, \Re[\alpha], \quad \langle\alpha|\mathcal{P}|\alpha\rangle = \sqrt{2m\hbar\omega} \, \Im[\alpha].$$

(d) Prove the following results for the means square position and momentum,

$$\langle\alpha|\mathcal{Q}^2|\alpha\rangle = \langle\alpha|\mathcal{Q}|\alpha\rangle^2 + \frac{\hbar}{2m\omega}, \quad \langle\alpha|\mathcal{P}^2|\alpha\rangle = \langle\alpha|\mathcal{P}|\alpha\rangle^2 + \frac{\hbar m\omega}{2},$$

and infer from them the minimum uncertainty $\Delta Q \Delta P = \frac{1}{2}\hbar$.

Solution: