[lex184] Phase-space distribution in harmonic oscillator

The time evolution of the canonical coordinates of the classical harmonic oscillator can be written (for a particular choice of phase) in the form,

$$x(t) = x_0 \sin(\omega t), \quad p(t) = p_0 \cos(\omega t), \quad \frac{1}{2} m\omega^2 x_0^2 = \frac{p_0^2}{2m} = E.$$

The time averaged joint probability density for the scaled coordinates $\hat{x} \doteq x/x_0$ and $\hat{p} \doteq p/p_0$ is uniform on the unit circle:

$$\bar{P}(\hat{x}, \hat{p}) = \frac{1}{2\pi} \delta\big(\sqrt{\hat{x}^2 + \hat{p}^2} - 1\big). \tag{1}$$

(a) Show that this probability density is indeed normalized.
(b) Show that integration of (1) over the variable $\hat{p}$ yields the probability density $P_x(\hat{x})$ identified in [lam5] and calculated in [lex182].

Solution: