[lex182] Position distribution in harmonic oscillator

The goal of this exercise is to connect the position distributions of the quantum and classical harmonic oscillators.

(a) The wave functions of the quantum harmonic oscillator in one spatial dimension are well known to be expressible by Hermite polynomials and Gaussian functions as follows:

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} \exp\left(-\frac{m\omega x^2}{2\hbar}\right) \mathrm{H}_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right), \quad n = 0, 1, 2, \ldots$$

The quantity $|\psi_n(x)|^2$ then represents the position distribution in the stationary state of energy level n. Plot the quantity $\sqrt{n}|\psi_n(\bar{x})|^2$ versus $\bar{x}/\sqrt{n}$ for $n = 25$, where $\bar{x} \doteq x/\sqrt{2\hbar/m\omega}$ is the (dimensionless) scaled position variable.

(b) The time evolution of the position coordinate of the classical harmonic oscillator can be written (for a particular choice of phase) in the form $x(t) = x_0 \sin(\omega t)$, where the square of the amplitude x_0 grows proportional to the energy. Calculate the classical distribution for the scaled position $\hat{x} \doteq x/x_0$ from the expression,

$$P(\hat{x}) = \frac{2}{\tau} \int_{-\tau/4}^{+\tau/4} dt\, \delta\big(\hat{x} - \sin(\omega t)\big), \quad \tau = \frac{2\pi}{\omega}.$$

(c) Explain why a rescaling of the quantum result with the factor $\sqrt{n}$ (as carried out) is necessary to provide a matching range and (local average) with the classical result.

Solution: