[lex181] Poisson distribution from maximizing uncertainty

In the range of super-Poisson statistics, for which the variance $\langle\langle n^2\rangle\rangle$ of the probability distribution $P_n(t)$ exceeds the mean $\langle n\rangle$ as discussed in [lln24], the Pascal distribution (representing thermal light of one wavelength) produces the highest uncertainty in the count of photons and the Poisson distribution (representing coherent light) produces the lowest uncertainty, where the uncertainty is defined as follows:

$$\Sigma(t) \doteq -\sum_{n=0}^{\infty} P_n(t) \ln P_n(t).$$

Nevertheless, the Poisson statistics can be obtained from a principle of maximum uncertainty, namely for the (continuous) probability distribution $f(t)$ of time intervals between photon detections in a stream of photons with average time interval τ. Show that the uncertainty,

$$\Sigma_f \doteq -\int_0^{\infty} dt f(t) \ln\big(f(t)\big),$$

is maximized for the functional form $f(t) = e^{-t/\tau}/\tau$, from which the Poisson probability distribution is inferred recursively as follows [lln24]:

$$P_0(t) = \int_t^{\infty} dt' f(t'), \quad P_n(t) = \int_0^t dt' f(t') P_{n-1}(t-t').$$

Solution: