[lex177] Super-Poisson interpolation statistics I

(a) Construct the super-Poisson probability distibution,

$$P_n(t) = \left(1+\frac{1}{m}\right)\left(1+\frac{2}{m}\right)\cdots\left(1+\frac{n-1}{m}\right)\frac{1}{n!}\left(\frac{t}{\tau}\right)^n\left(1+\frac{t}{m\tau}\right)^{-(m+n)},$$

from the function,

$$P_0(t) = \left(1+\frac{t}{m\tau}\right)^{-m}$$

and the differential recursion relation,

$$(n+1)P_{n+1}(t) = nP_n(t) - tP_n'(t), \quad n = 1, 2, \ldots$$

The integer-valued m is the interpolation parameter which links the Pascal distribution ($m = 1$) with the Poisson distribution ($m = \infty$). The function $P_0(t)$ is known to represent the well-established Pascal and Poisson distributions in the two limits [lln24] and the recursion relation has been demonstrated in [lex179] to be valid for both.
(b) Recover the limiting cases,

$$\lim_{m\to 1} P_n(t) = \frac{(t/\tau)^n}{(1+t/\tau)^{n+1}}, \qquad \lim_{m\to\infty} P_n(t) = \frac{(t/\tau)^n}{n!}e^{-t/\tau},$$

from the general expression stated above.

Solution: